Open problems in the theory of germs of analytic maps from n-space to $n+1$-space

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The theory of germs of maps $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ has an embarassing gap dating back nearly thirty years, concerning the relation between the rank of the vanishing homology of such a germ (the homology of the image of a stable perturbation) and its deformation-theoretic codimension, the $\mathcal{A}_e$-codimension. The former is known as the *image Milnor number*, and denoted by $\mu_I(f)$. There is a conjectural relation between the two invariants:

$$\mathcal{A}_e - \text{codim}(f) \leq \mu_I(f) \tag{0.1}$$

with equality if f is weighted homogeneous. It is known that this holds if $n = 1$ or 2 (see [4], [3]), but although there are plentiful examples suggesting that it holds in higher dimensions, the general conjecture remains unproven.

However, the geometry of stable perturbations is only poorly described by the image Milnor number. Consideration of low dimensional examples reveals a rich range of ways in which the image can acquire homology – see e.g. Figure 3 in [5]. It turns out that this behaviour can be described with the help of the *multiple point spaces* $D^k(f_t)$, defined as the closure of the set of k-tuples of pairwise distinct points sharing the same image under f_t. A homology theory which takes into account the action of the symmetric group Σ_k on $D^k(f_t)$ was developed in [2] and [1] for this purpose, and goes under the name of *alternating homology.* It gives a concise description: n-cycles in the image originate as $(n-k+1)$-cycles in the alternating homology of the multiple point spaces $D^k(f_t)$, for $2 \leq k \leq n+1$.

This mini-course will describe the geometry of finite-codimension map germs $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ and their stable perturbations, with a focus on examples outside the range of dimensions in which the conjecture has been proved correct. It will build up the necessary theoretical tools ($\mathcal{A}_e$- tangent space, multiple point spaces, Fitting ideals, alternating homology) to make a reasonably detailed analysis of the examples, and thus to understand their vanishing homology. It will compare the results with a formula for the image Milnor number of weighted homogeneous germs in the case $n = 3$ recently obtained by T. Ohmoto (and first presented at São Carlos, see [6]) using Thom polynomial techniques. It will emphasise open problems and seek to prepare students to tackle them.

References

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