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Equisingularity and the Theory of Integral Closure

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Introduction

"Let me now take a new tack which promises a better wind. Instead of dealing with a pair of hypersurfaces, let us consider analytic families of hypersurfaces V_r, all having a singular point at the origin and depending on a set of parameters." O. Zariski, Presidential Address, Bulletin A.M.S. 77 No. 4 (1971), 481-491 [23].

Given a family of sets or maps, when are all the members the same? When are some of the members different? Equisingularity is the study of these questions. As Zariski noticed it is easier to say when a member of family is different than it is to say when two sets or two maps are the same. Often the change in a single invariant suffices to pick out the members which are out of step with the rest.

A basic question is what do we mean by "the same"? And how do we tell when a family of sets are the same using invariants of the members of the family? These questions are explored in these notes.

As Zariski indicates earlier in his address, equisingularity had its roots in both differential topology and algebraic geometry, and both areas continue to contribute important ideas. The use of algebraic geometry naturally leads to the use of commutative algebra to count and to control.

In answering the question of what "the same " means a topologist might ask the members of the family be homomorphic; a differential topologist would ask that some of the infinitesimal structure, such as limiting tangent planes and secant lines be preserved as well, while an algebraic geometer might ask that the singularities have the same multiplicity.

In these notes we work in the complex analytic case using the Whitney conditions or Verdier's W, known to be equivalent in the complex analytic case, to say when the members of a family are the same. These conditions imply all three of the above possible answers. The theory of integral closure of ideals and modules provides an algebraic description of these conditions from which we may abstract the invariants which control them in families.

Here is an overview of my current approach to equisingularity questions. Given a set X, decide on the landscape that the set is part of. This means deciding on the allowable families that include the set, and the generic elements that appear in allowable families. Each set should have a unique generic element that it deforms to, and some elements of the topology of this generic element should be important invariants of our set. Describing the connection between the infinitesimal geometry of X and the topology of the generic element related to X is part of understanding the landscape. Based on the allowable deformations, determine the corresponding first order infinitesimal deformations of X. These make up a module $N(X)$. The Jacobian module of X, $JM(X)$ is the module generated by the partial derivatives of a set of defining equations for X. For the case of sets, the invariants we need for checking condition W come from the pair $(JM(X), N(X))$ and $N(X)$ by itself. A change at the infinitesimal level of the family is always tied to a change in topology of the generic related elements.

Those who have studied maps using stabilizations will recognize many elements of the overview in that context.

These notes are divided into three lectures with an afterword. They are designed to help you reach the point where the overview makes sense. In the afterword we will look at the overview again, using determinantal singularities as an example.

There are many exercises scattered through the notes. I encourage you to try all of them. There are also some readings which fill in gaps in the proofs or provide deeper understanding. I encourage you to try these as well as time permits.

A first reading which gives an overview of how the material in these notes developed can be found on the conference web site, along with the abstract for the course. It is a PDF of the talk I gave at Aussois in June '15 to celebrate the 70th birthday of Bernard Teissier. Teissier has made all of his papers available on his web site, (webusers.imj-prg.fr/ bernard.teissier/articles-Teissier.html) and many of the suggested readings can be found there.

1 Equisingularity Conditions

We start with some notation to describe a family of sets. In the diagram:

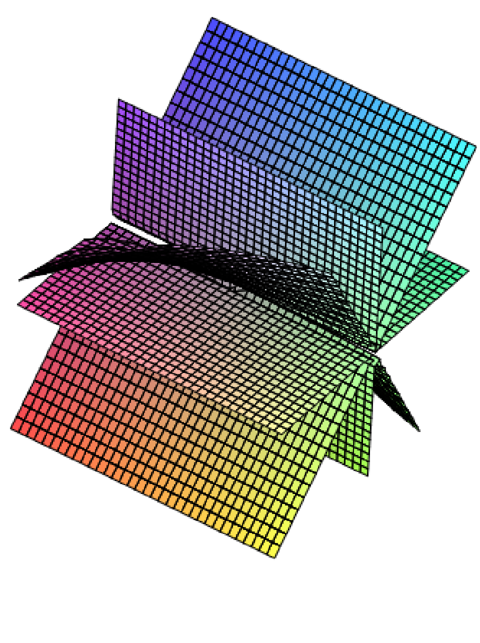
$$\begin{array}{ccccc} X^d(0) \subset & \mathcal{X}^{d+k} & \subset & Y \times \mathbb{C}^N \\ \downarrow & \downarrow p_Y & \swarrow \pi_Y & \\ 0 \in & Y = \mathbb{C}^k & & \end{array}$$

the parameter space is Y, $X(0)$ denotes the fiber of the family over $\{0\}$, $\mathcal{X}^{d+k}$ denotes the total space of the family which is contained in $Y \times \mathbb{C}^N$. We usually assume $Y \subset \mathcal{X}^{d+k}$, and $\mathcal{X} = F^{-1}(0)$, $X(y) = {f_y}^{-1}(0)$, where $f_y(z) = F(y, z)$.

Given a family of map germs as above, we say the family is holomorphically trivial if there exists a holomorphic family of origin preserving bi-holomorphic germs r_y such that $r_y(X(0)) = X(y)$. If the map-germs are only homeomorphisms we say the family is C^0 trivial.

Every subject needs a good example to start with. Here is ours:

Example 1.1. *Let $\mathcal{X}$ be the family of four moving lines in the plane with equation $F(x, y, z) = xz(z+x)(z-(1+y)x) = 0$. Here y is the parameter, the x and z axis are fixed as is the line $z+x=0$ while the line $z-(1+y)x=0$ moves with y. Here is a picture of the total space of the family:*



This family is not holomorphically trivial as the next exercise shows, but it should be equisingular for any reasonable definition of equisingularity.

Problem 1.2. *Show that the family of 4 lines is not homomorphically trivial by following the hints and proving them: If r_y is a trivialization of the family of sets, $Dr_y(0)$ must carry the tangent lines of $X(0)$ to $X(y)$. If a linear map preserves the lines defined by $x = 0, z = 0, z = -x$ then the linear map must be a multiple of the identity. Hence r_y can't map $z = x$ to $z = (1 + y)x$, $y \neq 0$.*

So we need a notion of equisingularity that is less restrictive than holomorphic equivalence.

The Whitney conditions imply C^0 triviality but also imply the family is well-behaved at the infinitesimal level.

If X is an analytic set, X_0 the set of smooth points on X, Y a smooth subset of X, then the pair (X_0, Y) satisfies **Whitney's condition A** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0::

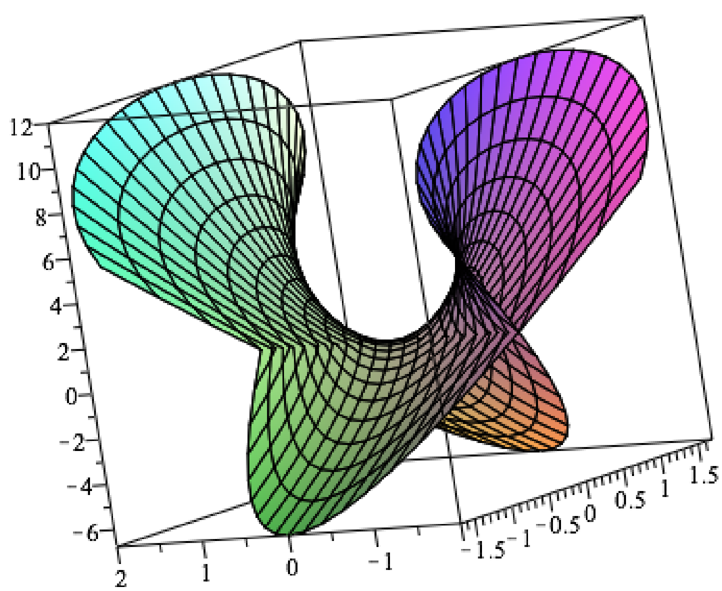
$$\begin{array}{c}\{x_i\} \to y \\ \{TX_{x_i}\} \to T\end{array} \Rightarrow T \supset TY_y$$

The pair (X_0, Y) satisfies **Whitney's condition B** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0:

$$\begin{array}{c}\{x_i\} \to y \\ \{TX_{x_i}\} \to T \\ sec(x_i, \pi_Y(x_i)) \to L\end{array} \Rightarrow T \supset L$$

Problem 1.3. *Show that the family of 4 lines satisfies the Whitney conditions. (Hint: The family consists of submanifolds meeting pairwise transversely.)*

Example 1.4. *This is a famous example used in many singularities talks. $\mathcal{X}$ is defined by $F(x, y, z) = z^3 + x^2 - y^2z^2 = 0$. The members of the family $X(y)$ consist of node singularities where the loop is pulled smaller and smaller as y tends to zero becoming a cusp at $y = 0$. Here is a picture:*



The singular locus is the y-axis. Whitney A holds because every limiting tangent plane contains the y-axis. But Whitney B fails. Notice that the parabola $z = y^2$ is in the surface, and letting $x_i = (0, t_i, {t_i}^2)$ and $y_i = (0, t_i, 0)$, t_i any sequence tending to 0, we see that the limiting secant line is the $z$axis, while the limiting tangent plane along this curve is the xy-plane.

We see that the dimension of the limiting tangent planes at the origin is 1, while it is zero elsewhere on the y-axis. This kind of drastic change at the infinitesimal level is prevented by the Whitney conditions.

Reading You can read about the Whitney conditions in many places. Two references are the first chapter of [5], and Chapter III of [22]. The latter is more in the spirit of the way we are developing the subject, though harder.

Verdier's condition W

The next condition, while equivalent to the Whitney conditions in the complex analytic case (proved by Teissier [22]) is easier to work with using algebra.

Condition W says that the distance between between the tangent space to X at a point x_i of X_0 and the tangent space to Y at y goes to zero as fast as the distance between x_i and Y. We first need to define what we mean by the distance between two linear spaces.

Suppose A, B are linear subspaces at the origin in $\mathbb{C}^N$, then define the distance from A to B as:

$$\operatorname{dist}(A, B) = \sup_{\substack{u \in B^{\perp} - \{0\} \\ v \in A - \{0\}}} \frac{\|(u, v)\|}{\|u\| \, \|v\|}.$$

In the applications B is the "big" space and A the "small" space. The inner product is the Hermitian inner product when we work over $\mathbb{C}$. The same formula also works over $\mathbb{R}$.

Example 1.5. *Work in $\mathbb{R}^3$. Let $A = x$-axis, B a plane with unit normal u_0, then the distance from A to B is $\cos\theta$, where θ is the small angle between u_0 and the x-axis, in the plane they determine. So when the distance is 0, B contains the x-axis.*

We recall Verdier's condition W.

Definition 1.6. *Suppose $Y \subset \bar{X}$, where X, Y are strata in a stratification of an analytic space, and* $\operatorname{dist}(TY_0, TX_x) \leq C\operatorname{dist}(x, Y)$ *for all x close to Y. Then the pair (X, Y) satisfies* **Verdier's condition** *W at $0 \in Y$.*

Problem 1.7. *Show that W fails for Teissier's example for X_0, Y where Y is the y-axis at the origin.*

We would like to re-write this condition in terms of F where F defines a hypersurface. This will allow us to develop an algebraic formulation of the W condition.

Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 1.8. *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0,0)$ if and only if there exists U a neighborhood of $(0,0)$ in $\mathcal{X}$ and $C > 0$ such that*

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y, z)\|$$

for all $(y, z) \in U$ and for $1 \leq l \leq k$.

Proof. In this set-up Y is a k-plane, so we will set $A = Y$, and calculate the distance between Y and a tangent plane to $\mathcal{X}_0$ at (y, z) which is our B. At a smooth point of $\mathcal{X}^{k+n}$, we can use $\overline{DF(y, z)}/\|DF(y, z)\|$ for $u \in B^{\perp}$, and the standard basis for the vectors from A.

Then the distance formula says that condition W holds if and only if

$$\sup_{1 \leq l \leq k} \frac{\|\frac{\partial F}{\partial y_l}(y, z)\|}{\|DF(y, z)\|} \leq C'' \mathrm{dist}((y, z), Y) = C' \sup_{1 \leq i \leq n+1} \|z_i\|$$

This is equivalent to

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{1 \leq i \leq n+1} \|z_i\| \sup_{1 \leq j \leq n+1} \|\frac{\partial F}{\partial z_j}(y, z)\|$$

From which the desired result follows.

□

Denote the ideal generated by the partial derivatives of F with respect to the z variables by $J_z(F)$, and the ideal generated by z_j by m_Y Then $z_i \frac{\partial F}{\partial z_j}$ are a set of generators for $m_Y J_z(F)$. The inequality above says that the partial derivatives of F with respect to y_l go to zero as fast as the ideal $m_Y J_z(F)$. We will examine the implications of this in the next section.

Reading After you read a little about the integral closure of ideals, reading p589-605 [20] will give you a good background on the integral closure approach to Whitney equisingularity for hypersurfaces with isolated singularities.

2 The Theory of Integral Closure of Ideals and Modules

Many operations on ideals and submodules of a free module come from an operation on rings.

We illustrate this idea by reviewing the notions of the integral closure of a ring and the normalization of an analytic space.

Definition 2.1. *Let A, B be commutative Noetherian rings with unit, $A \subset B$ a subring. Then $h \in B$ is integrally dependent on A if there exists a monic polynomial $f(T) = T^n + \sum_{i=0} f_i T^i$, $f_i \in A$ such that $f(h) = 0$. The integral closure of A in B consists of all elements of B integrally dependent on A.*

Example 2.2. *Let A be the ring of convergent power series in t^2, t^3 denoted $\mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. Then if $f(T) = T^2 - t^2$ we have $f(t) = 0$, so t is integrally dependent on A. In fact, B is the integral closure of A.*

Definition 2.3. *Let A be the local ring of an analytic space X, x, B the ring of meromorphic functions on X at x; the space associated with the integral closure of A in B is the normalization of X.*

Example 2.4. *Let $A = \mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. Then A is the local ring at the origin of the cusp $x^3 - y^2 = 0$, and since $t^3/t^2 = t$, the ring of meromorphic functions on X at the origin is $\mathbb{C}\{t\}$. So by the previous example the normalization of the cusp is a line.*

In this context a ring A is normal if the integral closure of A in its quotient field is A. A space germ is normal if its local ring is normal. Normal spaces have nice properties–they are non-singular in codimension 1 and the Riemann removable singularities theorem is true for them. Given a space germ X we always have a map π_{NX} from the normalization of X, denoted NX, to X which is finite and generically 1-1. NX and π_{NX} are unique up to holomorphic right equivalence. You can read proofs of these facts in [15] p 154-163, working backwards as necessary.

The following exercise is easy assuming the facts in the last paragraph.

Problem 2.5. *Show that the normalization of an irreducible curve germ X, x is $\mathbb{C}, 0$.*

If you know a little bit about singularities of maps the next exercise is also easy.

Problem 2.6. *Suppose $f : \mathbb{C}^n, 0 \to C^p, 0$, $n < p$ and f is a finitely determined map-germ. Show $(\mathbb{C}^n, 0), f$ is a normalization of the image of f.*

Basic Results from the Theory of Integral Closure for Ideals

The operation of integral closure of rings creates, as we shall see, an operation on ideals, the operation of forming the integral closure of I, which is an ideal, denoted $\overline{I}$. Assume I is an ideal in $\mathcal{O}_{X,x}$, $f \in \mathcal{O}_{X,x}$. In discussing the properties of integral closure, sometime we work on a small neighborhood of X. In this case, I refers to the coherent sheaf I generates on U.

List of Basic Properties f is integrally dependent on I if one of the following equivalent conditions obtain:

(i) There exists a positive integer k and elements a_j in I^j, so that f satisfies the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $\mathcal{O}_{X,0}$.

(ii) There exists a neighborhood U of 0 in $\mathbb{C}^N$, a positive real number C, representatives of the space germ X, the function germ f, and generators $g_1, \ldots, g_m$ of I on U, which we identify with the corresponding germs, so that for all x in U we have: $\|f(x)\| \leq C \max\{\|g_1(x)\|, \ldots, \|g_m(x)\|\}$.

(iii) For all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$ the pull–back $\phi^* f$ is contained in the ideal generated by $\phi^*(I)$ in the local ring of $\mathbb{C}$ at 0. If for all paths $\phi^* f$ is contained in $\phi^*(I)m_1$, then we say f is strictly dependent on I and write $f \in I^\dagger$.

Let NB denote the normalization of the blowup of X by I, $\bar{D}$ the pullback of the exceptional divisor of the blowup of X by I. Then we have:

(iv) For any component C of the underlying set of $\bar{D}$, the order of vanishing of the pullback of f to NB along C is no smaller than the order of the divisor $\bar{D}$ along C. This implies that the pullback of f lies in the ideal sheaf generated by the pullback of I.

The first property is usually taken as the definition, and shows that integral dependence is an algebraic idea. The second property is used to control equisingularity conditions. The third property is convenient for computations. The notion of strict dependence defined in the third property is used to describe properties like Whitney A, where integral dependence is insufficient–cf problem later on about Whitney A.

Given a curve $\phi(s)$, if $f \circ \phi$ is defined, it is equal to cs^r+ h.o.t. We call r the order of f on ϕ and write $f_\phi = r$, and J_ϕ for the order of an ideal J on ϕ.

Because the exceptional divisor of the blow-up of the Jacobian ideal tracks limiting infinitesimal information, the fourth property is perhaps the most important. Since NB is normal, each component of the exceptional divisor is generically a smooth submanifold of a manifold, so the ideal vanishing on the component is locally principal. This means we can talk about the order of vanishing on each component. The order of the divisor $\bar{D}$ is just the order of vanishing along the component of the pullback of I to NB. Concretely, pick a local generator u of the ideal of the component, and write the elements of I in terms of u. The smallest power of u that appears is the order of I along C.

The fourth property also shows how a closure operation on rings gives a closure operation on ideals–

start with a ring and an ideal, enlarge the ring by a closure operation, look at the ideal generated in the new ring, then intersect with the original ring to define the closure operation on the ideal.

Reading For detailed proofs of the equivalences between these properties see [18] p 18-27. You can download this paper from Teissier's list of publications–it is #15. Try this after reading the proofs of the equivalences contained here.

In the next example, we practice using the first property.

Example 2.7. *Let* $A = \mathcal{O}_2$*,* $I = (x^n, y^n)$*. Suppose* $f = x^i y^j$*,* $i+j \geq n$*. Consider the monic polynomial* $h(T) = T^n - (x^n)^i (y^n)^j$*. Since* $(x^n)^i (y^n)^j$ *is in* I^n*, and* $h(f) = 0$*,* $f \in \overline{I}$*.*

Now we do a computation using the third property.

Example 2.8. *Let* $A = \mathcal{O}_2$*,* $I = (x^a, y^b)$*. Given* $m = x^i y^j$ *define the weight* m *to be* $bi + aj$*, given* $f(x, y)$*, define the weight of* f *to be the minimum weight of all monomials appearing in a power expansion of* f*. We will show that* $\overline{I}$ *consists of all* f *such that weight of* $f > ab$*.*

First we'll show weight of $m \geq ab$ implies $m \in \overline{I}$. It suffices to check this for curves $\phi(t) = (t^r, t^s)$ as higher order terms don't affect the order of I or m on the curve. Since $\overline{I}$ is an ideal this will show that $f \in \overline{I}$.

We have I_ϕ =min$\{ra, sb\}$; assume $ra \leq sb$.

It is convenient to think of the monomial $x^i y^j$ as the point (i, j) in the xy-plane. Consider the parallel lines $rx + sy = c$. Then if m is any monomial on this line, $m_\phi = c$, and $m_\phi > c$ if m lies above this line. If the weight of $m \geq ab$ then m lies above or on the line connecting $(a, 0)$ and $(0, b)$, so it will lie above or on any line passing through $(a, 0)$, which lies below or on $(0, b)$. This implies that $m_\phi \geq ra$ and shows $m \in \overline{I}$.

Suppose the power expansion of f contains a monomial m which lies below the line connecting $(a, 0)$ and $(0, b)$. Then the convex hull of the monomials appearing in f has a vertex m' which lies below the line connecting $(a, 0)$ and $(0, b)$. We can find a line passing through this vertex which lies below $(a, 0)$ and $(0, b)$. Then for the curve ψ defined by this line,

$$f_\psi = m'_\psi < I_\psi$$

which shows that $f \notin \overline{I}$.

This kind of reasoning is very useful in studying properties of ideals which are well connected to their Newton polygons. In this example the Newton polygon of I is all the points of $\mathbb{R}^2$ above or on the line connecting $(a, 0)$ and $(0, b)$ in the first quadrant. For more examples and details see [21], which is #46 on Teissier's publication list or [19].

Next, we use property 2 to characterize Verdier's W in the hypersurface case.

Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 2.9. *Condition W holds for* $(\mathcal{X}_0, Y)$ *at* $(0, 0)$ *if and only if* $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ *for* $1 \leq l \leq k$*.*

Proof. By the last proposition of the first section and the following remarks we know that W holds if and only if

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y, z)\|$$

But, by property 2 this is equivalent to $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$. □

If we have a curve ϕ on $\mathcal{X}^{k+n}$, $\phi(0) = 0$, and the image of ϕ in $\mathcal{X}_0^{k+n}$ except at 0, and $J(F)_\phi = r$ then we can calculate the limiting tangent hyperplane to $\mathcal{X}^{k+n}$ along ϕ as

$$\underset{s\to 0}{limit}(1/s^r)(DF(\phi(s)))$$

If $\frac{\partial F}{\partial y_l} \in \overline{J_z(F)}$ for $1 \leq l \leq k$, then the limiting plane is never vertical, but it does not necessarily contain Y.

Problem 2.10. *Show that if $\frac{\partial F}{\partial y_l}$ for $1 \leq l \leq k$ is strictly dependent on $J_z(F)$ then every limit of tangent planes along every curve ϕ not in $V(J_z(F))$ contains Y.*

Problem 2.11. *Show that if $\frac{\partial F}{\partial y_l}$ for $1 \leq l \leq k$ is strictly dependent on $J_z(F)$ then WA holds.*

We will prove a few of the implications showing the equivalence of the basic properties.

Proposition 2.12. *Property 1 implies property 3*

Proof. Let f satisfy the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1}f + a_k = 0$ in $\mathcal{O}_{X,0}$, and let $\phi : \mathbb{C}, 0 \to X, 0$. Choose $g \in I$ such that $g_\phi = I_\phi$. We may assume the image of ϕ does not lie in $V(I)$. Then

$$\frac{(f\circ\phi)^k}{(g\circ\phi)^k} + \frac{a_1\circ\phi}{(g\circ\phi)}\frac{(f\circ\phi)^{k-1}}{(g\circ\phi)^{k-1}} + \cdots + \frac{a_{k-1}\circ\phi}{(g\circ\phi)^{k-1}}\frac{(f\circ\phi)}{(g\circ\phi)} + \frac{a_k\circ\phi}{(g\circ\phi)^k} = 0$$

and $\frac{a_i\circ\phi}{(g\circ\phi)^i}$ is holomorphic for all i. Since $\mathcal{O}_1$ is normal, it follows that $\frac{(f\circ\phi)}{(g\circ\phi)}$ is holomorphic, hence $f\circ\phi \in \phi^*(I)$. □

Proposition 2.13. *Property 3 implies property 4*

Proof. We will only prove this for the case where $V(I) = 0$.

Consider the components $\{C_i\}$ of $\bar{D}$. Since NB is normal and the C_i have codimension 1, we can pick out points c_i on each C_i and curves $\tilde{\phi}_i$, such that $\tilde{\phi}_i(0) = c_i$, and $\tilde{\phi}_i$ is transverse to C_i. We can choose c_i so that $\pi_{NB}^*(I)$ vanishes only on C_i in a neighborhood of c_i, and the same is true for $f\circ\pi_{NB}$. If u_i defines C_i at c_i, then we have $f\circ\pi_{NB} = h_i {u_i}^{f_i}$, h_i a unit, while we can find local generators of $\pi_{NB}^*(I)$ of form ${u_i}^{f_i}$ where f_i is the order of I along C_i. Now $\pi_{NB}\circ\tilde{\phi}_i$ is a map from $\mathbb{C}, 0 \to X, 0$, since $\pi_{NB}(C_i) = 0$. (This is the reason for restricting to this case.) Hence if property 3 holds $f_i \geq I_i$ for all i. □

Proposition 2.14. *Property 4 implies property 2*

Proof. Choose a compact neighborhood U of 0, and consider its inverse image in NB. The inverse image must be compact as well. So, since $f\circ\pi_{NB} \in \pi_{NB} * (I)$, we can cover $\pi_{NB}^{-1}(U)$ with a finite number of sets and choose elements of I such that

$$\|f\circ\pi_{NB}(p')\| \leq C\max\{\|g_1\circ\pi_{NB}(p')\|, \ldots, \|g_m\circ\pi_{NB}(p')\|\}$$

holds on $\pi_{NB}^{-1}(U)$. Then it is clear that

$$\|f(\pi_{NB}(p'))\| \leq C\max\{\|g_1(\pi_{NB}(p'))\|, \ldots, \|g_m(\pi_{NB}(p'))\|\}.$$

Since π_{NB} surjects on U this finishes the proof. □

We have Prop 2.9 to describe W for hypersurfaces, but what about higher codimension?

If $X = F^{-1}(0)$ where $F\ \mathbb{C}^n \to \mathbb{C}^p$, then at a smooth point p of X, the rowspace of the matrix of partial derivatives of F span the conormal space of X, p. Since the vectors of the conormal space are what we need to control the distance between the tangent space of X, p and $TY, 0$, this suggests we should look at the module generated by the partial derivatives of F denoted $JM(X)$, just as we looked at $J(F)$ in the hypersurface case.

Basic Results from the Theory of Integral Closure for Modules

Notation: $M \subset N \subset F^p$, F^p a free $\mathcal{O}_{X,x}$ module of rank p, M, N submodules of F. If M is generated by g generators $\{m_i\}$, then let $[M]$ be the matrix of generators whose columns are the $\{m_i\}$.

We will develop properties for modules similar to those for ideals; however a convenient entry way into the theory is:

Definition 2.15. *If $h \in F^p$ then h is integrally dependent on M, if for all curves ϕ, $h \circ \phi \in \phi^*(M)$. The integral closure of M denoted $\overline{M}$ consists of all h integrally dependent on M.*

A good very basic reference on properties of integral closure of modules is [6] p301-307.

Problem 2.16. *$\overline{M}$ is a module, $\overline{\overline{M}} = \overline{M}$*

Example 2.17. *Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, then $\overline{M} = m_2\mathcal{O}_2^2$.*

It is clear that $\overline{M} \subset m_2\mathcal{O}_2^2$; we will show that $\begin{pmatrix} y \\ 0 \end{pmatrix} \in \overline{M}$.

Given a curve ϕ we can assume $y_\phi < x_\phi$ otherwise $\begin{pmatrix} y \circ \phi \\ 0 \end{pmatrix} \in \begin{pmatrix} x \circ \phi \\ 0 \end{pmatrix} \mathcal{O}_1$.

Then

$$\begin{pmatrix} y \\ 0 \end{pmatrix} \circ \phi = \begin{pmatrix} y \\ x \end{pmatrix} \circ \phi - x/y \circ \phi \begin{pmatrix} 0 \\ y \end{pmatrix} \circ \phi$$

where $x/y \circ \phi \in \mathcal{O}_1$.

Connection with the theory of integral closure of ideals I

Notation: Given an element $h \in F$ and a submodule M, then (h, M) denotes the submodule generated by h and the elements of M. Given a submodule N of F, $J_k(N)$ denotes the ideal generated by the set of k by k minors of a matrix whose columns are a set of generators of N.

Theorem 2.18. *(Jacobian principle) Suppose the rank of (h, M) is k on each component of (X, x). Then $h \in \overline{M}$ if and only if $J_k(h, M) \subset \overline{J_k(M)}$*

Proof. The complete proof appears in [6] p304. We will show that $h \in \overline{M}$ implies $J_k(h, M) \subset \overline{J_k(M)}$, which is the easy part.

We have

$$\phi^*(J_k(h, M)) = J_k(\phi^*(h, M)) = J_k(\phi^*(M) = \phi^*(J_k(M))$$

which implies the result. $\square$

As an application we can develop the analogue of property 2 for ideals.

Proposition 2.19. *([6], Prop 1.11) Suppose $h \in \mathcal{O}^p_{X,x}$, M a submodule of $\mathcal{O}^p_{X,x}$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $\psi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$,*

$$\|\psi(z) \cdot h(z)\| \leq C \sup_i \|\psi(z) \cdot s_i(z)\|$$

for all $z \in U$.

For each choice of ψ, the $\{\psi \cdot s_i(z)\}$ give a linear combination of the rows of $[M]$ at each point, while $\psi(z) \cdot h(z)$ is the analogous combination of the entries of h. So the inequality of the theorem relates the size of row vectors of $[M(x)]$ to corresponding combinations of the entries of h.

Proof. We will use the Jacobian principle to show that the inequality implies the integral closure inclusion, by using special ψ_i.

Let S_I be a $k \times (k-1)$ submatrix of $[M]$, going through all such submatrices as I varies, let h_I be a k-tuple gotten by dropping the same entries from h as rows from $[M]$ in forming S_I Let $\psi_I(z)(h(z)) = \det[h_I(z), S_I(z)]$. Note that $psi_I(z)s_i(z) = \det[s_i(z), S_I(z)]$, a generator of $J_k(M)$.

The inequality which we are assuming then shows that $J_k(h, M) \subset \overline{J_k(M)}$, which gives the result by the Jacobian principle.

□

There is a useful variant of this Proposition.

Proposition 2.20. *([13] For a section $h \in \mathcal{O}^p_X$ to be integrally dependent on M at 0, it is necessary that, for all maps $\phi\,(\mathbb{C}, 0) \to (X, 0)$ and $\psi\,(\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$ with $\lambda \neq 0$, the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belong to the ideal $\psi(M \circ \phi)$.*

Conversely, it is sufficient that this condition obtain for every ϕ whose image meets any given dense Zariski open subset of X.

As an application we can extend our criterion for condition W to equidimensional sets of any codimension.

Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ an equidimensional family of equidimensional sets, $\mathcal{X}^{k+n} \subset Y^k \times \mathbb{C}^N$.

Proposition 2.21. *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0,0)$ if and only if $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$.*

Proof. Assume condition W holds at $(0,0)$. Suppose there exists some $\psi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$ such that

$$\|\psi(p) \cdot \frac{\partial F}{\partial y_l}(p)\| \leq C \sup_{(i,j)} \|\psi(p) \cdot z_j \frac{\partial F}{\partial z_i}(p)\|$$

fails on any neighborhood of the origin for any C for some y_j.

Then we can use $\psi(p) \cdot DF(p)$ to find a sequence of conormal vectors for which the inequality

$$\frac{\|(\psi(p) \cdot DF(p), e_l\|}{\|\psi(p) \cdot DF(p)\|} \leq C \sup_{1 \leq i \leq N} \|z_i\|$$

fails for any U and any C. But this implies W fails at the origin.

In [6] the opposite implication is proved by constructing a subspace of TX of dimension k which converges to $TY, 0$ at the desired rate. We will give a different proof by tracking limits using a blowing up construction, which we develop next.

□

Blowing up modules

We want to develop the analogue of property 4 for ideals but now for modules. We will want a construction that works for pairs of submodules, not just a single submodule.

Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. This is known as the Rees algebra of M. If $(m_1, \ldots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then $\operatorname{Projan}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal. Denote the projection to X^d by c, or by c_M where there is ambiguity.

Example 2.22. *If M is the Jacobian module of X, then* $\operatorname{Projan}(\mathcal{R}(M))$ *is $C(X)$, the projectivised conormal space of X.*

If M is a submodule of N or h is a section of N, then h and M generate ideals on $\operatorname{Projan}\mathcal{R}(N)$; denote them by $\rho(h)$ and $\mathcal{M}$. If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i/T_1$.

Example 2.23. *If M is the Jacobian module of X and $N = F^p$ then $V(\mathcal{M})$ consists of pairs (x, L) where $x \in X$ and $L \in \mathbb{P}Hom(\mathbb{C}^p, \mathbb{C})$, and $L \circ DF(x) = 0$.*

Using Proposition 2.20 it is easy to show that h is integrally dependent on M at the origin, if and only the ideal sheaf induced from h is integrally dependent as an ideal sheaf on $\mathcal{M}$ along $0 \times \mathbb{P}^{p-1}$. This is the second connection between integral closure of ideals and modules.

Looking at a pair (M, N) allows us to "strip out" one copy of N from M, as the following example shows.

Example 2.24. *Let $M = I = (x^2, xy, z) = J(z^2 - x^2y)$ and $N = J = (x, z)$. M is the Jacobian ideal of the Whitney umbrella, and N defines the singular locus of the umbrella. So, working on $\mathbb{C}^3$* $\operatorname{Projan}\mathcal{R}(N) = B_J(\mathbb{C}^3$, *which has ring $R = \mathbb{C}[T_1, T_2]/(zT_1 - xT_2)$, and where the map from $\mathcal{R}(N)$ to R is given by $x \to T_1, z \to T_2$. Writing the generators of I in terms of the generators of J as $x^2 = x \cdot x, xy = y \cdot x, z = z$ the map from $\mathcal{R}(I)$ to R has image (xT_1, yT_1, T_2) and this induces the ideal sheaf $\mathcal{I}$ on* $\operatorname{Projan}\mathcal{R}(N)$. *We see that this is supported only at the point* $(0, [1, 0])$.

Having defined the ideal sheaf $\mathcal{M}$, we blow up by it. The advantages of this we will see in the next section, as it gives a constructive/geometric way to calculate the multiplicity of a pair of modules. But for now, this gives the context for which property 4 in the ideal case holds. As an example of how the blow up comes up, if we are in the basic set-up, and $M = m_Y JM(\mathcal{X})$ then the blow up by $\mathcal{M}$ is the blowup of the conormal of $\mathcal{X}$ by the ideal defining the stratum Y. Teissier has shown that condition W holds for the pair $(\mathcal{X}_0, Y)$ at the origin iff and only if the exceptional divisor of this blow up is equidimensional over Y. We will see the proof of one direction of this in the next section as well.

3 Multiplicities, Integral closure and the Multiplicity-Polar Theorem

The multiplicity of an ideal or module or pair of modules is one of the most important invariants we can associate to an m-primary module. It is intimately connected with integral closure. It has both a length theoretic definition and intersection theoretic definition. We give the definition in terms of length first, for ideals, and submodules of a free module

Theorem/Definition 3.1. *(Buchsbaum-Rim [2]) Suppose $M \subset F$, M, F A-modules, F free of rank p, A a Noetherian local ring of dimension d, F/M of fine length, $\mathcal{F} = A[T_1, \ldots, T_p]$, $\mathcal{R}(M) \subset \mathcal{F}$, Then*

$\lambda(n)$ =length$\mathcal{F}_n/\mathcal{M}_n$ is eventually a polynomial $P(M,F)$ of degree d+p-1.

Writing the leading coefficient of $P(M,F)$ as $e(M)/(d+p-1)!$, then we define $e(M)$ as the multiplicity of M.

It is possible to compute simple ideal examples by hand as we show:

Example 3.2. *Let $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. Then $e(M) = 4$.*

Here is why. We have $p = 1$, $F = \mathcal{O}_2$, and we work with $\mathcal{F} = \mathcal{O}_2[T_1]$. (Notice that Projan $\mathcal{F} = \mathbb{C}^2$.) Now $\mathcal{M}_n = I^nT^n = m_2^{2n}T^n$, so

$$l(\mathcal{F}_n/\mathcal{M}_n) = l(\mathcal{O}_2/m^{2n} = (2n)(2n+1)/2 = 4n^2/2! + (l.o.t.)$$

So $e(M) = 4$.

Problem 3.3. *Let $M = I = (x^2, y^2) \subset \mathcal{O}_2$. Show $e(M) = 4$. (Hint: Try to show that the terms that are missing in this problem due to the missing xy term, grow only linearly with n, so the leading term of the polynomial is the same.)*

It is possible to do the very simplest module examples by hand easily as well.

Problem 3.4. *Let $M = m_2\mathcal{O}_2^2$. Show $e(M) = 3$.*

The next problem is harder–try to use the same strategy as in Problem 3.3.

Problem 3.5. *Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. Show $e(M) = 3$.*

If $\mathcal{O}_{X^d,x}$ is Cohen-Macaulay, and M has $d+p-1$ generators where $M \subset F^p$, then there is a useful relation between M and its ideal of maximal minors and the multiplicity of both of them. The multiplicity of M is the colength of M, and is also the colength of the ideal of maximal minors, by some theorems of Buchsbaum and Rim [2], 2.4 p.207, 4.3 and 4.5 p.223. A proof of this theorem in the context of analytic geometry using the Multiplicity Polar theorem is given in [11] Using this result, it is easy to do Problem 3.5.

Challenge Problem 3.6. *Buchsbaum and Rim showed $e(M) = l(F^p/M)$, if M has d+p−1 generators, F a module over a Cohen-Macaulay ring. What is a generalization of this to $e(M,N)$? (If M and N are ideals there is something along these lines in [10] Theorem 2.3.)*

An important theorem both for computational and theoretical purposes was proved by Rees in the ideal case. A proof of a generalization to modules appears in [16]

Theorem 3.7. *Suppose $M \subset N$ are m primary submodules of F^p, and $\overline{M} = \overline{N}$. Then $e(M) = e(N)$. Suppose further that $\mathcal{O}_{X,x}$ is equidimensional, then $e(M) = e(N)$ implies $\overline{M} = \overline{N}$*

Several generalizations of this result exist: Kleiman and Thorup proved a result for pairs of modules involving three modules, where the multiplicity of each pair was defined [16] . Generalizations also exist where the multiplicity is not defined. Gaffney and Gassler did the case of ideals [12], and Gaffney for modules [9], while Ulrich and Valadoshti have an approach using the epsilon multiplicity.

For computational purposes, this is coupled with another result–given any $M \subset F^p$, M a module over a local ring of dimension d, there exists a submodule R of M with $d+p-1$ generators such that $\overline{M} = \overline{R}$. Such an R is called a *reduction* of M.

So if $\mathcal{O}_{X^d,x}$ is Cohen-Macaulay, we can try to find a reduction R of M with the right number of generators $d+p-1$, then calculate the length of F/R. (This length is also called the colength of R.) Here is a very simple example.

Problem 3.8. *Suppose I is any ideal in $\mathcal{O}_2$ which contains x^n, y^n and all other terms have degree at least n. Then $e(I) = n^2$.*

Now we want to give an intersection theoretic definition of the multiplicity. This definition applies to pairs of modules as well.

The next diagram shows the spaces that come into the definition.

$$\begin{array}{ccc} B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N)) & \xrightarrow{\pi_N} & \operatorname{Projan}\mathcal{R}(N) \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_{XN}} \\ \operatorname{Projan}\mathcal{R}(M) & \xrightarrow{\pi_{XM}} & X \end{array}$$

On the blow up $B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N))$ we have two tautological bundles. One is the pullback of the bundle on $\operatorname{Projan}\mathcal{R}(N)$. The other comes from $\operatorname{Projan}\mathcal{R}(M)$. Denote the corresponding Chern classes by c_M and c_N, and denote the exceptional divisor by $D_{M,N}$. Suppose the generic rank of N (and hence of M) is g.

Then the multiplicity of a pair of modules M, N is:

$$e(M,N) = \sum_{j=0}^{d+g-2} \int D_{M,N} \cdot c_M^{d+g-2-j} \cdot c_N^j.$$

Kleiman and Thorup show that this multiplicity is well defined at $x \in X$ as long as $\overline{M} = \overline{N}$ on a deleted neighborhood of x. This condition implies that $D_{M,N}$ lies in the fiber over x, hence is compact. Notice that when $N = F$ and M has finite colength in F then $e(M,N)$ is the Buchsbaum-Rim multiplicity $e(M, \mathcal{O}^p_{X,x})$.

Let's re-calculate two examples using this definition.

Example 3.9. *Let $M = I = (x^2, xy, y^2) \subset \mathcal{O}_2$. Then $e(M) = 4$.*

Here $d = 2$, $p = g = 1$, $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2$, $\operatorname{Projan}\mathcal{M}) = B_I(\mathbb{C}^2) = B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N))$, and $\operatorname{Projan}\mathcal{M}) \subset \mathbb{C}^2 \times \mathbb{P}^1$. So the only term we need to calculate is $\int D_{M,N} \cdot c_M$. We can calculate this term as follows: Intersect $B_I(\mathbb{C}^2)$ with $\mathbb{C}^2 \times H$, H a generic hyperplane in $\mathbb{P}^1$, which represents $c(M)$. Project this curve to $\mathbb{C}^2$, and calculate the order of I on the curve. Projecting the curve to $\mathbb{C}^2$ amounts to setting a generic combination of the generators to zero, and looking at the curve obtained, removing any components in $V(I)$. In this case a generic curve is $x^2 - ay^2 = 0$, $a \neq 0$. This consists of two branches ($x - y = 0$ and $x + y = 0$ if $a = 1$) and the colength of the ideal on each branch is 2 so the multiplicity is $2 + 2 = 4$.

Problem 3.10. *Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. Show $e(M) = 3$.*

Here $d = 2$, $p = g = 2$, $N = \mathcal{O}_2^2$, $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$, $\operatorname{Projan}\mathcal{M}) \subset \mathbb{C}^2 \times \mathbb{P}^2$, dimension of $B_{\mathcal{M}}(\operatorname{Projan}\mathcal{R}(N))$ is 3. So we need to calculate $\int D_{M,N} \cdot c_M^2$, $\int D_{M,N} \cdot c_M \cdot c_N$ (Notice that $c_N^2 = 0$, since we are working on $\operatorname{Projan}\mathcal{R}(N) = \mathbb{C}^2 \times \mathbb{P}^1$.) Now we have two choices: as before we intersect a representative of each class with the blow-up then push down to X, then see what the multiplicity of M is on each curve. Or, we can push down to $\operatorname{Projan}\mathcal{R}(N)$ and evaluate $\mathcal{M}$ on each curve.

Taking the second route, projecting the intersection of the blow-up with a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^1$ and a hyperplane from $\mathbb{C}^2 \times \mathbb{P}^2$, is a curve on $\mathbb{C}^2 \times \mathbb{P}^1$, defined by a linear relation $T_1 = aT_2$, and by setting one of the elements of $\mathcal{M}$ restricted to this set to zero. The restriction of $\mathcal{M}$ to the locus $T_1 = aT_2$ is the ideal generated by the entries of the linear combination of the first row and a times the second row from the original matrix. A generic curve is given by setting $x + ay = 0$, and the multiplicity of $\mathcal{M}$ on this curve is 1. So, $\int D_{M,N} \cdot c_M \cdot c_N = 1$.

Projecting the intersection of the blow-up with two hyperplanes from $\mathbb{C}^2 \times \mathbb{P}^2$, amounts to setting two generic elements of $\mathcal{M}$ to zero and removing any components of $V(\mathcal{M})$. Setting $xT_1 + yT_2$ and $yT_1 + xT_2 = 0$ gives two curves. One curve is $x = y$, $T_1 = 1 = -T_2$ and the other curve is $x = -y$, $T_1 = 1 = T_2$.

The restriction of $\mathcal{M}$ to the first curve is x so the multiplicity is 1; as it is on the second curve as well, for a total of 3.

Notice that in the last example $3 = e(M) \neq e(J(M)) = 4$. (J(M) is the ideal of maximal order non-vanishing minors, and is (x^2, xy, y^2) in this case.) But,

Problem 3.11. *Suppose $M \subset N \subset F$ are m primary $\mathcal{O}_{X,x}$ modules, X, x equidimensional. Show that $e(M) = e(N)$ if and only if $e(J(M)) = e(J(N))$*

There are examples though, where there is a family of ICIS singularities where $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not. In the example due to Henry and Merle, the embedding dimension of the singularity changes at $y = 0$–the singularity goes from being codimension 2 to being codimension 1, because one of the defining equations is no longer singular off the origin. Is this the only way for the connection between the two invariants to break?

Challenge Problem 3.12. *Give a geometric characterization of when $e(JM(X_y))$ is independent of y, but $e(J(JM(X_y)))$ is not.*

Reading In section 3 of [8] these ideas are developed further. It also contains the example due to Henry and Merle mentioned above.

Polar Varieties of a Module

Intuitively, the polar varieties of a module measure the "curvature" of Projan $\mathcal{R}(M)$, and we have encountered them in the computations of the previous paragraph. As we shall see, the projection of $B_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N)) \cdot c_M^2$ to $\mathbb{C}^2$ is the polar curve of M.

The *polar variety of codimension* l of M in X, denoted $\Gamma_l(M)$, is constructed by intersecting Projan $\mathcal{R}(M)$ with $X \times H_{g+l-1}$ where H_{g+l-1} is a general plane of codimension $g+l-1$, then projecting to X.

The polar varieties of M can be constructed by working only on X. The plane H_{g+l-1} consists of all hyperplanes containing a fixed plane H_K of dimension $g + l - 1$; By multiplying the matrix of generators of M by a basis of H_K we obtain a submodule of M denoted M_H.

Proposition 3.13. *In this set-up the polar variety of codimension l consists of the closure in X of the set of points where the rank of M_H is less than g, and the rank of M is g.*

Proof. Since H_{g+l-1} is generic, the general point of Projan $\mathcal{R}(M) \cap X \times H_{g+l-1}$ lies over points where the rank of M is g. Choose coordinates so that a basis for H_K consists of the last $g + l - 1$ elements of the standard basis of $\mathbb{C}^j$, j the number of generators of M. We can find v such that $v[M_H] = 0$ but $v[M] \neq 0$ if and only if we are at a point where the rank of $M_H < g$. The existence of v is equivalent to being able to find a combination of the rows of $[M]$, such that the last $g + l - 1$ entries are 0. This row is a hyperplane which lies in H_{g+l-1}. □

There is a special case which will be important to us. The diagram below represents the smoothing of an isolated singularity.

$$\begin{array}{ccccc} X^d(0) \subset & \mathcal{X}^{d+1} \subset Y \times \mathbb{C}^N & \supset \mathcal{X}(y) \\ \downarrow & \downarrow{\scriptstyle p_Y} & \downarrow{\scriptstyle \pi_Y} \\ 0 \in & Y = \mathbb{C} & \supset y \neq 0 \end{array}$$

Let $M = JM_z(\mathcal{X})$, Then $\Gamma_d(\mathcal{X})$ by the previous proposition is defined by selecting $N-1$ generic generators of $JM_z(\mathcal{X})$, and looking to see where they have less than maximal rank. Assume coordinates chosen so that the first $N-1$ columns of $[JM(\mathcal{X})]$ are generic. Then the points where the polar intersects $\mathcal{X}(y)$ are the critical points of z_N restricted to $\mathcal{X}(y)$. The number of such points is the number of sheets of $\Gamma_d(\mathcal{X})$ over Y is the multiplicity of $\Gamma_d(\mathcal{X})$ over Y at the origin. It is clear that the number of critical points of a generic linear form on a smoothing of X is important to the topology of $\mathcal{X}(y)$, so this number is an important invariant of X.

There is a strong connection between polar varieties and integral closure thanks to an important result of Kleiman and Thorup [16], [17].

By definition the existence of a polar variety of M at $x \in X$ is tied to the dimension of the fiber of $\operatorname{Projan} \mathcal{R}(M)$ over x. The following theorem ties the dimension of this fiber to integral closure conditions.

Set-up: X the germ of a reduced analytic space of pure dimension d, F a free $\mathcal{O}_X$-module, $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d+e-1$. Set $C := \operatorname{Projan}(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset \mathcal{S}ym\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c\, C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Theorem 3.14. *(Kleiman-Thorup) If N is not integral over M, then E has dimension $r-1$, the maximum possible.*

We give an example the usefulness of this Theorem by giving a simple proof of one direction of a theorem of Teissier describing Whitney equisingularity.

Set-up: Suppose $Y^k, 0 \subset X^{d+k}, 0$, Y^k smooth, $\underline{y}$ coordinates on Y, $I(Y) = m_Y$. Set $M = m_Y JM(X)$, $N = M + \mathbb{C}\{\frac{\partial f}{\partial \underline{y}}\}$, then $Projan(\mathcal{R}(M)) = B_{m_Y}(C(X))$, $M = N$ off Y.

Let E denote the exceptional divisor of $B_{m_Y}(C(X))$.

Theorem 3.15. *(Teissier, [22]) If the fibers of E, the exceptional divisor of $B_{m_Y}(C(X))$ over Y, have the same dimension, then the Whitney conditions hold along Y.*

Proof. If the Whitney conditions fail along Y, they do so on a proper closed subset $S \subset Y$. Then S is the set where $\overline{M} \neq \overline{N}$. ([6]) By the K-T theorem there must be a component of E over S, so the fibers of E have larger dimension over points in S than over the generic point of Y. □

For ICIS, we use the multiplicity of $mJM(X)$ to control the dimension of the fibers of E. What do we do if the multiplicity is not defined? Try $e(JM(X), N)$, for N related to the geometry of X. So, we must relate the multiplicity of a pair to the polar varieties of M.

Set-up: $M \subset N \subset F$, a free $\mathcal{O}_X$ module, X equidimensional, a family of sets over Y,with equidimensional fibers, Y smooth, $\overline{M} = \overline{N}$ off a set C of dimension k which is finite over Y.

Let $\Delta(e(M,N)) = e(M(0), N(0), \mathcal{O}_{X(0)}, 0) - e(M(y), N(y), \mathcal{O}_{X(y)}, (y,x))$ be the change in the multiplicity of the pair (M,N) as the parameter changes from y to 0.

Theorem 3.16. *(Multiplicity Polar Theorem, [7], [8]):*

$$\Delta(e(M,N)) = mult_y \Gamma_d(M) - mult_y \Gamma_d(N)$$

Determinantal singularities

Given M, a $(n+k,n)$ matrix, with entries in $\mathcal{O}_q$; view M as a map from $\mathbb{C}^q \to Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$. Assume M is transverse to the rank stratification of $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ on $\mathbb{C}^q - 0$. Let $X_M := V(I_M)$, I_M generated by the maximal minors of M. X_M is determinantal i.e. $\operatorname{codim}(X_M)$ is as small as possible. If $q < 2(2+k)$ then X_M has a smoothing.

We fix the class of deformations and fix a unique smoothing by only considering deformations of X_M which come from deformations of the entries of M. Geometric meaning of invariants depends on the smoothing. We may freely vary the entries of M.

Deformations of the entries of M induce deformations of the generators of I; first order deformations define the module $N(X_M)$. Generators of $N(X_M)$ are tuples of minors of M of size $n-1$.

Properties of $N(X)$

The operation of forming $N(X)$ has some nice properties.

- N is universal. If the entries of M are coordinates on $Hom(\mathbb{C}^n, \mathbb{C}^{n+k})$ denote $N(X)$ by N_U. Then for any M, $N(X_M) = M^* N_U$.
- N_U is stable; $N_U = JM(\Sigma)$, Σ the matrices of less than maximal rank.
- Stability implies the polar varieties of Σ are the polar varieties of N_U.
- Universality implies $\Gamma_i(N(X_M)) = M^*\Gamma_i(N_U)$.
- Together they imply if $\tilde{M}$ defines a smoothing $\tilde{X}$ of X^d_M, then
-
$$mult_{\mathbb{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}})) = M(\mathbb{C}^q) \cdot \Gamma_d(\Sigma)$$

This last formula holds in fact for all determinantal singularities, not just those defined by maximal minors

Equisingularity of Determinantal Varieties

For maximal minors, $M(\mathbb{C}^q) \cdot \Gamma_d(\Sigma)$ is computed in terms of the entries of M in [14].

As a corollary of the MPT, $e(mJM(X_{M(y)}), N(X_{M(y)})) + M(y)(\mathbb{C}^q) \cdot \Gamma_d(\Sigma)$ controls the Whitney conditions for the open stratum of X_M along Y. The precise statement follows. The proof brings together all of the elements of our course.

Theorem 3.17. *Suppose* $(X^{d+k}, 0) \subset (\mathbb{C}^{n+k}, 0)$,

$X = F^{-1}(0)$, $F : \mathbb{C}^{n+k} \to \mathbb{C}^p$, *$Y$ a smooth subset of X, coordinates chosen so that* $\mathbb{C}^k \times 0 = Y$, *$F$ induced from a deformation of the presentation matrix of X_0, X equidimensional with equidimensional fibers, of expected dimension.*

A) Suppose X_y are isolated, maximal rank determinantal singularities, suppose the singular set of X is Y. Suppose $e_\Gamma(m_y JM(F_y), N(\mathcal{X})(y))$ is independent of y. Then the union of the singular points of X_y is Y, and the pair of strata $(X - Y, Y)$ satisfies condition W.

B) Suppose $\Sigma(X)$ is Y and the pair $(X - Y, Y)$ satisfies condition W. Then $e_\Gamma(m_y JM(F_y), N_D(y))$ is independent of y.

Proof. Now we prove A). We can embed the famiy in a restricted versal unfolding with smooth base $\tilde{Y}^l$. Consider the polar variety of $JM_z(F)$ of dimension l, and the degree of its projection to $\tilde{Y}^l$ along points of Y. The hypothesis on e_Γ implies by the multiplicity polar theorem that this degree is constant over Y. In turn this implies that the polar variety over Y does not split, hence the polar of the original deformation is empty. This implies that the fiber of the exceptional divisor of $B_{m_y}\,\mathrm{Projan}(JM_z(F))$ cannot be maximal, since there is no polar variety. By the theorem of Kleiman-Thorup on the dimension of this fiber, it then follows that $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ which implies W.

This also implies that $\overline{JM(F)} \subset \overline{JM_z(F)}$. Hence the union of the singular points of F_y which is the cosupport of $\overline{JM_z(F)}$ is equal to the cosupport of $\overline{JM(F)}$ which is Y. Then the inclusion $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ implies W for $(X-Y, Y)$. (Cf [6])

Now we prove B). W implies $JM_Y(F) \subset \overline{m_Y JM_z(F)}$ which implies that $\overline{m_Y JM(F)} = \overline{m_Y JM_z(F)}$. We know by [22] that condition W implies that the fiber dimension of the exceptional divisor of $B_{m_Y}(C(X))$ over each point of Y is as small as possible. The integral closure condition $\overline{m_Y JM(F)} = \overline{m_Y JM_z(F)}$ implies that the same is true for $B_{m_Y}(\mathrm{Projan}\,\mathcal{R}(JM_z(F)))$. This implies that the polar of $m_Y JM_z(F)$ is empty, hence by the multiplicity polar formula the invariant $e_\Gamma(m_y JM(F_y), N_D)(y))$ is independent of y. □

We also have a geometric description of our invariant based on the smoothing and the existence of a unique Milnor fiber.

Theorem 3.18. *$e(JM(X_{M(y)}), N(X_{M(y)})) + M(y)(\mathbb{C}^q)\cdot\Gamma_d(\Sigma) = (-1)^d\chi(X_{s,y}) + (-1)^{d-1}\chi((X\cap H)_{s,y}), X_{s,y}$ a smoothing of $X(y)$.*

Proof. (Cf. [8] p 130, [1].) □

Many Challenge Problems! (From a talk at the Luminy Winter School in Singularities 2015)

- Can we say something about the other Betti-numbers of a smoothing when there is more than 1? (Frühbis-Krüger and Zach have some results for three-folds. Cf "On the Vanishing Topology of Isolated Cohen-Macaulay Codimension 2 Singularities", Arxiv 2015.)
- What is the connection between our results and Damon-Pike [3] in the (2,3) case?
- What is the relation in the curve case between our results and those of Greuel and Buchweitz and Rosenlicht differentials?
- For what determinantal singularities is the invariant $m_d(X^d) = 0$? Hopefully we can classify them. (This one was done in May 2015)
- Is there a way to connect the terms that appear in the calculation of the multiplicity of the polar of N with the geometry of X?
- What other classes of singularities do these methods apply to?
- What can we say about EIDS? (These are determinantal singularities which are isolated, but cannot be smoothed.)
- There are other invariants associated with X such as the index of differential forms and the Milnor number(?) of functions with isolated singularities. Compute these.

4 Afterword: Examples of the Point of View of the Introduction

We will talk about two examples of our point of view.

Hypersurfaces with isolated singularities are our first example. Suppose $X^n, 0$ has an isolated singularity at the origin, $X = f^{-1}(0)$.

Choose the landscape This is done by looking at the possible deformations of X. We see we can deform f freely, and still, for small deformations, get a hypersurface with at most isolated singularities. So, the landscape will be all of hypersurfaces in $\mathbb{C}^{n+1}$ with at most isolated singularities. The generic element that X deforms to is its Milnor fiber.

Describe the connection between X and its generic element To do this deform X to its Milnor fiber, using $F(y,z) = f(z) - y$. Then the ideal $J_z(F)$, when restricted to the graph, vanishes only at $(0,0)$, so it's polar curve is given by the vanishing of the first n partial derivatives, in generic coordinates. Applying the MPT, we get $e(J(f), \mathcal{O}_{X,0}) = mult_{\mathbb{C}}\Gamma_n(J_z(F))$.

In turn $mult_{\mathbb{C}}\Gamma_n(J_z(F))$ is the colength of the ideal $(f, \frac{\partial f}{\partial z}_1, \ldots, \frac{\partial f}{\partial z}_n)$ in $\mathcal{O}_{n+1}$. This is $\mu(X) + \mu(X \cap H)$, H a generic hyperplane.

Determine the first order infinitesimal deformations Since $f \to f + tg$ where g is arbitrary, is a first order defomation, and the corresponding infinitesimal first order deformation is $f \to \frac{\partial f + tg}{\partial t} = g$, the first order infinitesimal deformations are just $\mathcal{O}_{X,0}$.

Our invariants for controlling Whitney equisingularity are $e(mJ(f), \mathcal{O}_{X,0})$.

If we have a family of hypersurfaces $\mathcal{X}$, then if $\mu(X) + \mu(X \cap H)$ changes, then so must $e(J(f), \mathcal{O}_{X,0})$, and the exceptional divisor of $B_{J_z(F)}(\mathcal{X})$ must pick up a vertical component and vice-versa. The change in the topology of the landscape is reflected in a dramatic change in the fibers of the exceptional divisor, which is the infinitesimal information.

For determinantal singularities the story is the same.

If we look at all possible deformations, then we have examples where the same singularity can be deformed in two different ways; so we restrict our deformations by using the same size presentation matrix. The entries of the matrix can be deformed freely.

So the landscape will be the determinantal singularities corresponding to a matrix of fixed size. the generic element associated to X will be smooth given some dimension restriction; otherwise we can say what the stabilizations of the singularity are, and can begin to study those.

In the case of smoothable singularities, by use of the multiplicity polar theorem and some topology we get Proposition 3.18 which gives the connection between the topology of smoothing and the algebraic invariants of the singularity which are connected to its infinitesimal geometry.

The first order infinitesimal deformations of X can be explicitly computed; deform an entry of the presentation matrix by t, calculate the minors of the order used to define X; taking derivative with respect to t then gives a map from defining equations for X into tuples in $\mathcal{O}^g_{X,0}$, where g is the number of defining equations. These give the generators of $N(X)$. It is clear from this formulation that N is universal and specializes well in families. We can calculate $JM(\Sigma)$ explicitly–the partial with respect to the (i,j) entry of the matrix is just the corresponding generator of N. So Σ is stable. This gives the formula for showing $mult_{\mathbb{C}}\Gamma_d(N(\tilde{X}_{\tilde{M}}))$ can be computed using the presentation matrix, but leaves the formula in terms of the entries still to be determined in general.

Once again, a change at the infinitesimal level of the family is always tied to a change in topology of the generic related elements.

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