# Topology of real singularities

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# Chapter 1

# Introduction

This mini-course is aimed at young researchers and graduate students who want to learn basic tools and techniques of real singularity theory.

It starts with well-known notions and results of differential topology: the Brouwer degree, the index of a vector-field, the Poincaré-Hopf theorem, Morse functions. Although theses notions may be very familiar to any researcher experienced in singularity theory, we believe it is worth recalling them here.

In the next chapter, we apply these techniques of differential topology to some real analytic or semi-analytic sets and we get several nice formulas for topological invariants of these sets. In Chapter 4, always using the same techniques, we give several methods for the computation of the Euler characteristic of a real algebraic affine set.

We end with a chapter about the real Milnor fibration and the topology of the real Milnor fibre.

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# Chapter 2

# Tools of differential topology

In this chapter, we give the main tools and results in differential topology that we will need and apply in the study of real singularities. Our main references are [20], [22], [28] and [29].

## 2.1 The Brouwer degree

Let $M$ and $N$ be two oriented $n$-dimensional manifolds (without boundary) and let $f : M \to N$ be a smooth (i.e. $C^\infty$) map. We assume that $M$ is compact and $N$ is connected. Let $x$ be a regular point of $f$. This means that $Df(x) : T_xM \to T_{f(x)}N$ is a linear isomorphism between oriented vector spaces. We define the "sign" of $Df(x)$ to be $+1$ (resp. $-1$) if $Df(x)$ preserves (resp. reverses) the orientation.

**Definition 2.1.1** *For any regular value $y \in N$, we define*

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign } Df(x).$$

**Remark 2.1.2** *Since $y$ is a regular value of $f$, $f^{-1}(y)$ consists of regular points. Since $M$ is compact, $f^{-1}(y)$ is a finite collection of points since it is a 0-dimensional manifold.*

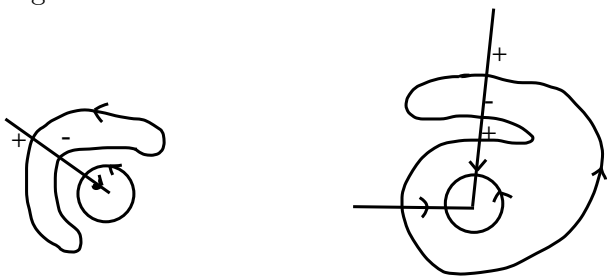
**Theorem 2.1.3** *The integer $\deg(f, y)$ does not depend on the choice of the regular value $y$.*

**Definition 2.1.4** *It is called the (Brouwer) degree of $f$, denoted by $\deg f$.*

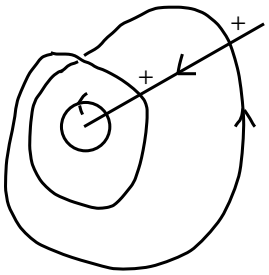
**Theorem 2.1.5** *If $f$ is smoothly homotopic to $g$ then $\deg f = \deg g$.*

For the proof of these two theorems, the reader is referred to Milnor's book [28].

Examples: 1) The map on the left has degree 0 and the map on the right has degree 1.




2) The map pictured below has degree 2.




3) Let $r_i : S^n \to S^n$ be defined by

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

The degree of $r_i$ is $-1$.

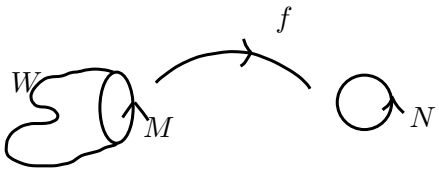
4) The map $S^1 \subset \mathbb{C} \to S^1 \subset \mathbb{C}$, $z \mapsto z^m$, $m \in \mathbb{Z}^*$, has degree $m$. Actually each point has exactly $m$ preimages and the map is regular and preserves orientation if $m > 0$ and reverses it if $m < 0$.

**Proposition 2.1.6** *If $M$, $N$ and $L$ are three smooth oriented manifolds of the same dimension, $M$ and $N$ are compact, $N$ and $L$ are connected and $f : M \to N$ and $g : N \to L$ are two smooth maps then* $\deg(g \circ f) = \deg g \times \deg f$.

*Proof.* It is easy from the definition. □

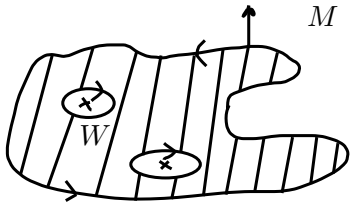
Example: The antipodal map $\sigma : S^n \to S^n$, $x \mapsto -x$ has degree $(-1)^{n+1}$ because $\sigma = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$. Hence if $n$ is even, $\sigma$ is not smoothly homotopic to the identity.

**Theorem 2.1.7** *If $M = \partial W$, $W$ is compact, and $f : M \to N$ extends to $F : W \to N$ then* $\deg f = 0$.




Application: Assume that $M$ is a smooth compact connected hypersurface in $\mathbb{R}^n$. By the Jordan-Brouwer Separation Theorem, it bounds a connected open set $D \subset \mathbb{R}^n$, i.e. $M = \partial\overline{D}$. This induces a natural orientation on $M$. Let $F : \overline{D} \to \mathbb{R}^n$ be a map that does not vanish on $\partial\overline{D} = M$. We assume that $F$ inside $D$ has a finite number of zeroes $p_1, \ldots, p_m$ that are all regular points of $F$. We can define $\overline{F} = \frac{F}{|F|} : M \to S^{n-1}$. Then we have

$$\deg \overline{F} = \sum_{i=1}^{m} \text{sign } \det[DF(p_i)].$$




Let us explain this equality. We remove a small open ball $B(p_i, \epsilon_i)$ around each $p_i$. Let $W = \overline{D} \setminus \cup_{i=1}^{m} B(p_i, \epsilon_i)$, it is a manifold with boundary

$$\partial W = M \bigcup \cup_{i=1}^{m} S(p_i, \epsilon_i).$$

The submanifold $\partial W$ is oriented by the canonical orientation of the boundary. Let us consider $\overline{F}_{\partial W} : \partial W \to S^{n-1}, x \mapsto \frac{F}{|F|}$. Then $\overline{F}_{\partial W}$ extends to $W$, so $\deg \overline{F}_{\partial W} = 0$. On the other hand, we have

$$\deg \overline{F}_{\partial W} = \deg \overline{F} - \sum_{i=1}^{m} \deg \overline{F}_i,$$

where $\overline{F}_i = \frac{F}{|F|} : S(p_i, \epsilon_i) \to S^{n-1}$. A minus sign appears here because the orientation of $S(p_i, \epsilon_i)$ as a component of the boundary of $W$ is the opposite of the orientation of $S(p_i, \epsilon_i)$ as the boundary of $B(p_i, \epsilon_i)$.

Since $p_i$ is a regular point of $F$, the degree of $\overline{F}_i$ is equal to the sign of $\det[DF(p_i)]$, because $\frac{F}{|F|} : S(p_i, \epsilon_i) \to S^{n-1}$ is homotopic to the map

$$S(p_i, \epsilon_i) \to S^{n-1}, p \mapsto \frac{DF(p_i)(p - p_i)}{|DF(p_i)(p - p_i)]|},$$

and therefore has the same degree as the map

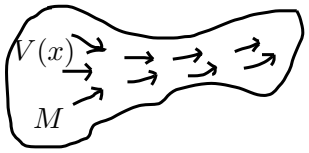
$$S(0, \epsilon_i) \to S^{n-1}, h \mapsto \frac{DF(p_i)(h)}{|DF(p_i)(h)|}.$$

This last map has degree equal to sign $\det [DF(p_i)]$, because the map $h \mapsto DF(p_i)(h)$ is homotopic to $\pm Id_{\mathbb{R}^n}$, depending on the sign of $\det[DF(p_i)]$ since $GL(n, \mathbb{R})$ has two connected components.

## 2.2 Vector fields and the Poincaré-Hopf theorem

**Definition 2.2.1** *Let $M$ be a smooth manifold. A vector field $V$ on $M$ is a smooth mapping $V : M \to TM$ such that for all $x \in M$, $pr(V(x)) \in T_xM$, where $pr : TM \to M$ is the natural projection.*

**Remark 2.2.2** *If $M \subset \mathbb{R}^n$ then a vector field is just a smooth mapping $V : M \to \mathbb{R}^n$ such that for all $x \in M$, $V(x) \in T_xM$.*




**Definition 2.2.3** *Let $p$ be an isolated zero of a vector field $V$ on a manifold $M$ of dimension $n$. In local coordinates, we can view $V$ as a mapping from a small open set $U \subset \mathbb{R}^n$ to a small open set $U' \subset \mathbb{R}^n$ where $0 \in U$ and $0 \in U'$ and such that $0$ is the only zero of $V$ in $U$. We define the Poincaré-Hopf index of $V$ at $p$ by*

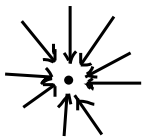
$$\text{Ind}(V, p) = \text{degree of } \frac{V}{|V|} : S^{n-1}_\epsilon \to S^{n-1},$$

*where $S^{n-1}_\epsilon$ is a small sphere centered at $0$ and included in $U$.*

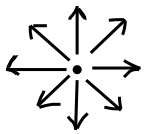
Examples in $\mathbb{R}^2$:

1. If $V(x,y) = (y,-x)$ (circulation) then $\mathrm{Ind}(V,0) = +1$.

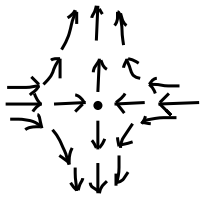
2. If $V(x,y) = (-x,-y)$ (sink) then $\mathrm{Ind}(V,0) = +1$.



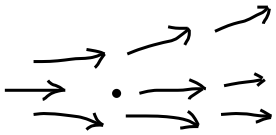
3. If $V(x,y) = (x,y)$ (source) then $\mathrm{Ind}(V,0) = +1$.



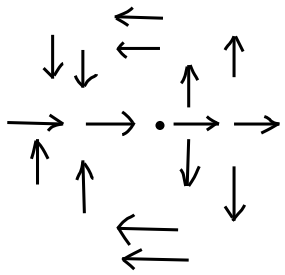
4. If $V(x,y) = (-x,y)$ (saddle) then $\mathrm{Ind}(V,0) = -1$.



5. If $V(x,y) = (x^2, x+y)$ then $\mathrm{Ind}(V,0) = 0$.



6. If $V(x,y) = (x^2 - y^2, 2xy)$ ($z \mapsto z^2$ in complex coordinates) then $\mathrm{Ind}(V,0) = +2$.



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**Remark 2.2.4** *One may check that this definition does not depend on the local coordinates (see [28] and [20] for example).*

**Theorem 2.2.5** *(Poincaré-Hopf theorem) Let $M$ be a smooth compact manifold. Let $V$ be a smooth vector field on $M$, with a finite number of zeroes $p_1, \ldots, p_k$. Then we have*

$$\chi(M) = \sum_{i=1}^{k} \mathrm{Ind}(V, p_i).$$

*Proof.* See [28], [20] or [22]. □

## 2.3 Morse functions

**Definition 2.3.1** *Let $M$ be a smooth manifold of dimension $n$, let $p \in M$ and $f : M \to \mathbb{R}$ be a smooth function. Let $(x_1, \ldots, x_n)$ be local coordinates around $p$ in $M$. We say that $p$ is a non-degenerate critical point of $f$ if $p$ is a critical point of $f$ (i.e. $\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0$) and the matrix*

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]_{1 \leq i,j \leq n},$$

*is non-singular.*

**Remark 2.3.2** *One may check that this definition does not depend on the choice of the local coordinate system (see [20]).*

**Proposition 2.3.3** *(Morse lemma) Let $p \in M$, $\dim M = n$, be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$. There exists a local coordinate system $(u_1, \ldots, u_n)$ around $p$ such that*

$$f = f(p) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2.$$

*Proof.* See [29]. □

**Definition 2.3.4** *The integer $\lambda$ is called the index of $f$ at $p$.*

**Corollary 2.3.5** *Non-degenerate critical points are isolated (in the set of critical points).*

**Definition 2.3.6** *Let $M$ be a smooth manifold. A function $f : M \to \mathbb{R}$ is a Morse function if it admits only non-degenerate critical points.*

**Theorem 2.3.7** *(Openness and density) For any manifold $M$, Morse functions form a dense open set in $C^\infty_s(M, \mathbb{R})$ (Whitney strong topology).*

*Proof.* It is a consequence of Thom's transversality theorem (see [22], [19] or [1]). □

For our applications, we will be mainly interested in fibres of analytic or polynomial mappings, so from now on, we shall assume that $M \subset \mathbb{R}^N$ and that dim $M = n$. Let $f : M \to \mathbb{R}$ be a smooth function. This defines a vector field $\nabla_M f$ (the gradient vector field of $f$ on $M$) by

$$\forall p \in M, \forall v \in T_pM, Df(p)(v) = \langle \nabla_M f(p), v \rangle.$$

Hence $p$ is a critical point of $f$ if and only if $\nabla_M f(p) = 0$. If $p$ is a Morse critical point of $f$ of index $\lambda$, then there is a local coordinate system such that

$$f = f(p) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2,$$

and so

$$\nabla_M f = (-2u_1, \ldots, -2u_\lambda, 2u_{\lambda+1}, \ldots, 2u_n).$$

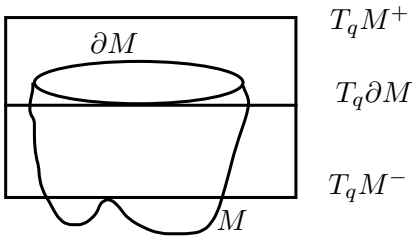
Hence the Poincaré-Hopf index $\mathrm{Ind}(\nabla_M f, p)$ is equal to $(-1)^\lambda$ because, as already explained above, the mapping $\frac{\nabla_M f}{|\nabla_M f|} : S(p, \epsilon) \to S^{n-1}$ has degree equal to sign $\det[D(\nabla_M f)(p)]$.

**Theorem 2.3.8** *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. Then, we have*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)},$$

*where $\lambda(p_i)$ is the Morse index of $p_i$.*

We will also consider the case of manifolds with boundary. Let $(M, \partial M) \subset \mathbb{R}^N$ be a manifold with boundary. Let $q \in \partial M$, then $T_q \partial M$ is a hyperplane in $T_q M$ and $T_q M = T_q \partial M \sqcup T_q M^+ \sqcup T_q M^-$ where $T_q M^+$ consists of outwards pointing vectors (outward vectors for short) and $T_q M^-$ consists of inwards pointing vectors.



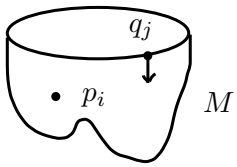

**Definition 2.3.9** *Let $q \in \partial M$ and let $f : (M, \partial M) \to \mathbb{R}$ be a smooth function. We say that $q$ is a correct critical point of $f$ if $q$ is a critical point of $f_{|\partial M} : \partial M \to \mathbb{R}$ and $Df(q)_{|T_q M}$ is not identically zero.*

**Definition 2.3.10** *We say that $f : (M, \partial M) \to \mathbb{R}$ is a correct Morse function if $f$ admits only Morse critical points on $M \setminus \partial M$ and $f_{|\partial M}$ admits only Morse correct critical points.*

**Theorem 2.3.11** *Let $(M, \partial M) \subset \mathbb{R}^N$ be a compact manifold with boundary and let $f : M \to \mathbb{R}$ be a correct Morse function. Denote by $p_1, \ldots, p_k$ the critical points of $f_{|M \setminus \partial M}$ and by $q_1, \ldots, q_l$ those of $f_{|\partial M}$. Then we have*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)} + \sum_{j \mid \nabla_M f(q_j) \text{ inward}} (-1)^{\mu(q_j)},$$

*where $\lambda(p_i)$ is the Morse index of $f$ at $p_i$ and $\mu(q_j)$ is the Morse index of $f_{|\partial M}$ at $q_j$.*




In the following chapters of this mini-course, we will use relative versions of the previous two theorems on Morse theory.

**Theorem 2.3.12** *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. For any $\alpha \in \mathbb{R}$, we have:*

$$\chi(M \cap \{f \geq \alpha\}, M \cap \{f = \alpha\}) = \sum_{i \mid f(p_i) > \alpha} (-1)^{\lambda(p_i)},$$

*where $\lambda(p_i)$ is the Morse index of $p_i$.*

Theorem 2.3.11 has a similar relative version.

# Chapter 3

# The Eisenbud-Levine formula and the Khimshiashvili formula

## 3.1 The Eisenbud-Levine formula

As seen in the second chapter, the Poincaré-Hopf index of a vector field plays an important role in the topology of manifolds. Here we present an algebraic formula for this index.

Let $f = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a $C^\infty$ map-germ (this is exactly the local expression of a vector field on a smooth manifold). We assume that

$$Q(f) = \frac{C^\infty(\mathbb{R}^n, 0)}{(f_1, \ldots, f_n)},$$

is a finite dimensional vector space over $\mathbb{R}$. Here $C^\infty(\mathbb{R}^n, 0)$ is the algebra of germs at $0 \in \mathbb{R}^n$ of $C^\infty$ real valued functions and $(f_1, \ldots, f_n)$ is the ideal generated by the components $f_1, \ldots, f_n$ of $f$. We write $\dim_{\mathbb{R}} Q(f) < +\infty$. We denote by $J_f$ the jacobian of the map-germ $f$. Namely, we have

$$J_f = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

**Theorem 3.1.1** *(The Eisenbud-Levine formula) Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a $C^\infty$ map-germ such that $\dim_{\mathbb{R}} Q(f) < +\infty$. Then we have*

1. *$0$ is isolated in $f^{-1}(0)$,*
2. *$J_f \neq 0$ in $Q(f)$,*

3. $\forall g \in Q(f)$, $gJ_f = g(0)J_f$ *in* $Q(f)$,

4. *let* $\varphi : Q(f) \to \mathbb{R}$ *be a linear form such that* $\varphi(J_f) > 0$ *and let* $\Phi : Q(f) \times Q(f) \to \mathbb{R}$ *be the bilinear symmetric form defined by* $\Phi(g,h) = \varphi(gh)$. *Then* $\Phi$ *is non-degenerate and* signature $\Phi = \mathrm{Ind}(f,0)$.

*Proof.* See [18], [1] or [3]. For a first approach, see [17]. □

Example: Let $f$ be the map-germ defined by

$$\begin{array}{rccc} f & : & (\mathbb{R}^2,0) & \to & (\mathbb{R}^2,0) \\ & & (x,y) & \mapsto & (x^2-y^2, 2xy). \end{array}$$

We have $Q(f) = \frac{C^\infty(\mathbb{R}^2,0)}{(x^2-y^2,2xy)}$. We see that $\dim_{\mathbb{R}} Q(f) = 4$ and that $\bar{1}$, $\bar{x}$, $\bar{y}$ and $\overline{x^2+y^2}$ form a basis of $Q(f)$. It is clear that 0 is isolated in $f^{-1}(0)$. Let us compute $J_f$:

$$J_f(x,y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2).$$

Let $\varphi : Q(f) \to \mathbb{R}$ be the linear form given by

$$\varphi(\bar{1}) = \varphi(\bar{x}) = \varphi(\bar{y}) = 0 \text{ and } \varphi(\overline{x^2+y^2}) = \frac{1}{4}.$$

Then $\varphi(J_f) = 1$. Let $\Phi$ be the linear symmetric form defined by $\Phi(P,Q) = \varphi(PQ)$. Let us compute its matrix in the basis $(\bar{1}, \bar{x}, \bar{y}, \overline{x^2+y^2})$. We have

$$\Phi(\bar{1},\bar{1}) = \varphi(\bar{1}) = 0, \Phi(\bar{1},\bar{x}) = \Phi(\bar{x},\bar{1}) = 0, \Phi(\bar{1},\bar{y}) = \Phi(\bar{y},\bar{1}) = 0,$$

$$\Phi(\bar{x},\bar{x}) = \varphi(\bar{x}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{y},\bar{y}) = \varphi(\bar{y}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{x},\bar{y}) = \varphi(\overline{xy}) = \varphi(\bar{0}) = 0 = \Phi(\bar{y},\bar{x}),$$

$$\Phi(\bar{1},\overline{x^2+y^2}) = \frac{1}{4}, \Phi(\bar{x},\overline{x^2+y^2}) = \varphi(\bar{x}^3 + \overline{xy^2}) = \varphi(\bar{0}) = 0,$$

$$\Phi(\bar{y},\overline{x^2+y^2}) = 0, \Phi(\overline{x^2+y^2},\overline{x^2+y^2}) = \varphi\big(\overline{(x^2+y^2)(x^2+y^2)}\big) = 0.$$

So this matrix is

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\frac{1}{8}$ with multiplicity 2, $\frac{1}{4}$ with multiplicity 1 and $-\frac{1}{4}$ with multiplicity 1. So the signature of $\Phi$ is $3-1=2=\mathrm{Ind}(f,0)$.

The Eisenbud-Levine formula gives an algebraic formula for the index of a vector field, hence an algebraic and "effective" way to compute a topological data. In the sequel, using technics introduced in the second chapter, we will present several formulas relating topological invariants to indices of vector fields. Thanks to the Eisenbud-Levine formula, these topological invariants become algebraically computable.

## 3.2 The Khimshiashvili formula

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be an analytic-map germ with an isolated critical point at 0. The Khimshiashvili formula (see [24]) relates the Poincaré-Hopf index of the gradient vector field $\nabla f$ of $f$ to the topology of a small regular level of $f$.

**Theorem 3.2.1** *We have*

$$\chi\big(f^{-1}(\delta)\cap B_\epsilon^n\big)=1-\mathrm{sign}(-\delta)^n\mathrm{Ind}(\nabla f,0), \tag{1}$$

*where $\delta$ is a regular value of $f$, $0<|\delta|\ll\epsilon\ll 1$, and*

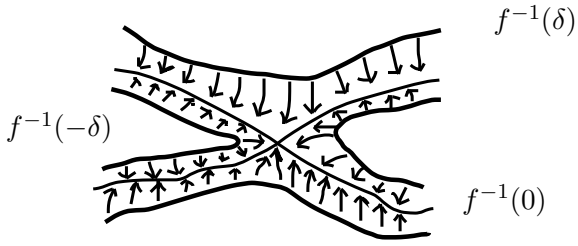
$$\chi(\{f\geq\delta\}\cap B_\epsilon^n)-\chi(\{f\leq\delta\}\cap B_\epsilon^n)=\mathrm{sign}(-\delta)^{n+1}\mathrm{Ind}(\nabla f,0). \tag{2}$$

*Proof.* Let $U$ be a small open subset of $\mathbb{R}^n$ such that $0\in U$, and $f$ is defined in $U$. We pertub $f$ into a Morse function $\tilde{f}:U\to\mathbb{R}$. Let $p_1,\ldots,p_k$ be the critical points of $\tilde{f}$, with respective indices $\lambda_1,\ldots,\lambda_k$. Let $\delta>0$, by Morse theory we have

$$\chi\big(f^{-1}([-\delta,\delta])\cap B_\epsilon^n\big)-\chi\big(f^{-1}(-\delta)\cap B_\epsilon^n\big)=\sum_{i=1}^{k}(-1)^{\lambda_i}.$$

Actually we can choose $\tilde{f}$ sufficiently close to $f$ so that the $p_i$'s lie in $f^{-1}([-\frac{\delta}{4},\frac{\delta}{4}])$. Now, $f^{-1}([-\delta,\delta])\cap B_\epsilon^n$ retracts to the central fibre $f^{-1}(0)\cap B_\epsilon^n$ and $f^{-1}(0)\cap B_\epsilon^n$ is the cone over $f^{-1}(0)\cap S_\epsilon^{n-1}$ (see [30]) so

$$\chi\big(f^{-1}([-\delta,\delta])\cap B_\epsilon^n\big)=1.$$



Moreover, we have

$$\sum_{i=1}^{k}(-1)^{\lambda_i} = \sum_{i=1}^{k} \text{sign det}[D(\nabla \tilde{f})(p_i)].$$

The sum on the right hand-side is the degree of the map $\frac{\nabla \tilde{f}}{|\nabla \tilde{f}|} : S_\epsilon^{n-1} \to S^{n-1}$ which is equal, by homotopy, to the degree of $\frac{\nabla f}{|\nabla f|} : S_\epsilon^{n-1} \to S^{n-1}$. By definition, this last degree is $\text{Ind}(\nabla f, 0)$. This gives the result for a negative regular value. For a positive regular value, we apply the result to $-f$ and use the relation $\text{Ind}(-\nabla f, 0) = (-1)^n \text{Ind}(\nabla f, 0)$. This proves formula (1). Formula (2) is proved with similar arguments. □

We will call $f^{-1}(\delta) \cap B_\epsilon^n$ the (positive or negative) real Milnor fibre. The following formulas are due to Arnol'd [2] and Wall [38].

**Corollary 3.2.2** *With the same hypothesis on $f$, we have*

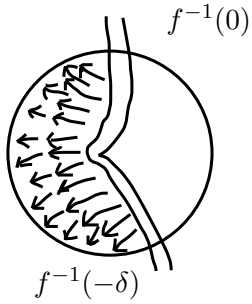
$$\chi(\{f \leq 0\} \cap S_\epsilon^{n-1}) = 1 - \text{Ind}(\nabla f, 0),$$

$$\chi(\{f \geq 0\} \cap S_\epsilon^{n-1}) = 1 + (-1)^{n-1} \text{Ind}(\nabla f, 0).$$

*If $n$ is even, we have:*

$$\chi(\{f = 0\} \cap S_\epsilon^{n-1}) = 2 - 2 \text{ Ind}(\nabla f, 0).$$

*Proof.* By a deformation argument due to Milnor [30], $f(-\delta) \cap B_\epsilon^n$, $\delta > 0$, is homeomorphic to $\{f \leq -\delta\} \cap S_\epsilon^{n-1}$, which is homeomorphic to $\{f \leq 0\} \cap S_\epsilon^{n-1}$ if $\delta$ is very small.




□

## 3.3 Non-isolated critical points

We would like to obtain similar results in the case when 0 is not an isolated critical point of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. The following result is due to Szafraniec [32].

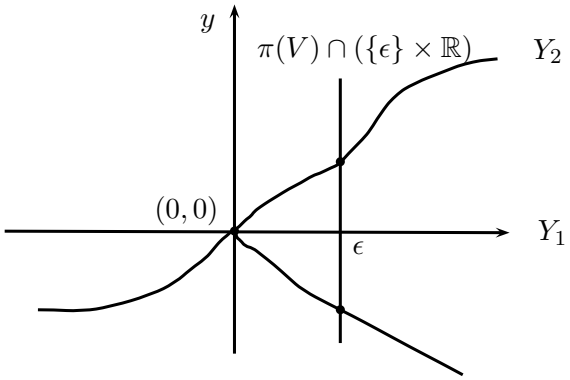
Let $\omega(x) = x_1^2 + \cdots + x_n^2$. We suppose that $f$ is defined in an open set $U$, $0 \in U$, of $\mathbb{R}^n$. Let $\Sigma_{f,\omega}$ be the following polar set:

$$\Sigma_{f,\omega} = \{x \in U \mid \operatorname{rank}(\nabla\omega(x), \nabla f(x)) \leq 1\},$$

and let $V$ be defined by

$$V = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid \omega(x) = \epsilon^2, x \in \Sigma_{f,\omega}, y = f(x)\}.$$

Then $V$ is an analytic subset of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Let $\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the projection $\pi(x, \epsilon, y) = (\epsilon, y)$. Then $\pi_{|V} : V \to \mathbb{R} \times \mathbb{R}$ is proper in the neighborhood of the origin. Hence $\pi(V)$ is closed and semi-analytic in the neighborhood of the origin. Let us set $Y_1 = \mathbb{R} \times \{0\}$ and $Y_2 = \overline{\pi(V) \setminus Y_1}$. The set $Y_2$ is semi-analytic.




If $\epsilon \neq 0$ then

$$\pi(V) \cap (\{\epsilon\} \times \mathbb{R}) = \{\epsilon\} \times \left\{\text{critical values of } f_{|S_\epsilon^{n-1}}\right\}.$$

So $\pi(V) \cap (\{\epsilon\} \times \mathbb{R})$ is finite and $\dim \pi(V) = \dim Y_2 = 1$. Therefore 0 is an isolated point of $Y_1 \cap Y_2$. By the Łojasiewicz's inequality [26], there exist $C > 0$ and $\alpha > 0$ such that

$$d\left((\epsilon, y), Y_1\right) \geq C d\left((\epsilon, y), \{0\}\right)^{2\alpha},$$

for $(\epsilon, y) \in Y_2$. Here $d((\epsilon, y), Z)$ is the distance from $(\epsilon, y)$ to the set $Z$. This implies that $|y| \geq C\epsilon^{2\alpha}$ for $(\epsilon, y) \in Y_2$ sufficiently close to $(0, 0)$ and that for $x \in \Sigma_{f,\omega} \setminus \{f = 0\}$ close to the origin,

$$|f(x)| \geq C\omega(x)^k, \qquad (*)$$

if $k \in \mathbb{N}$ and $k \geq \alpha$.

**Proposition 3.3.1** *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic function germ. Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be defined by $g = f - c\omega^k$ with $c \in ]0, C[$, $k \in \mathbb{N}$ and $k \geq \alpha$. Then $g$ has an isolated critical point at the origin. Moreover, for $0 < \epsilon \ll 1$,*

$$\chi(\{f \leq 0\} \cap S_\epsilon^{n-1}) = 1 - \mathrm{Ind}(\nabla g, 0).$$

*Proof.* We work in an open set $U$, $0 \in U$, as above. Note that $\Sigma_{f,\omega} = \Sigma_{g,\omega}$ in the neighborhood of the origin. Furthermore we see that $g$ does not vanish on $\Sigma_{g,\omega} \setminus \{0\}$ by the inequality $(*)$. Therefore if $\epsilon$ is small enough, 0 is a regular of $g_{|S_\epsilon^{n-1}}$. This implies that $g$ has an isolated critical point at the origin because $\{\nabla g = 0\} \subset \{g = 0\}$ and $\{\nabla g = 0\} \subset \Sigma_{g,\omega}$ in a neighborhood of the origin.

Let us fix $\epsilon > 0$ sufficiently small. Let $N_f = \{x \in S_\epsilon^{n-1} \mid f(x) \leq 0\}$ and

$$N_g = \{x \in S_\epsilon^{n-1} \mid g(x) \leq 0\} = \{x \in S_\epsilon^{n-1} \mid f(x) \leq c\epsilon^k\}.$$

Then $N_f \subset N_g$. Furthermore, $f_{|S_\epsilon^{n-1}}$ has no critical point on $N_g \setminus N_f$ by the inequality $(*)$ and so $N_f$ is a retract by deformation of $N_g$. But by Corollary 3.2.2,

$$\chi(N_g) = 1 - \mathrm{Ind}(\nabla g, 0).$$

□

Of course, applying this argument to $-f$, we obtain a similar formula for $\chi(\{f \geq 0\} \cap S_\epsilon^{n-1})$ and, as a consequence of the Mayer-Vietoris sequence, a formula for $\chi(\{f = 0\} \cap S_\epsilon^{n-1})$.

When $f$ is homogeneous, we can improve the previous result.

**Proposition 3.3.2** *( [6], [34]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree $d$. Let $k = d + 1$ if $d$ is odd or $k = d + 2$ if $d$ is even. Then the function $g$ defined by*

$$g(x) = f(x) - \frac{x_1^k + \cdots + x_n^k}{k},$$

*has an algebraically isolated critical point at* $0$ *(i.e.* $0$ *is an isolated zero of* $\nabla g_{\mathbb{C}}^{-1}(0)$*) and*

$$\chi\left(\{f \leq 0\} \cap S^{n-1}\right) = 1 - \mathrm{Ind}(\nabla g, 0).$$

**Remark 3.3.3** *Since* $0$ *is an algebraically isolated critical point of* $g$*, we can use the Eisenbud-Levine formula to compute* $\mathrm{Ind}(\nabla g, 0)$*, which was not necessarily the case in Proposition 3.3.1.*

# Chapter 4

# Degree formulas and signature formulas for the Euler characteristic of algebraic sets

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping and let $W = F^{-1}(0)$. We would like to compute $\chi(W)$ in terms of the polynomials $F_1, \ldots, F_k$. We will see that it is possible in some cases.

## 4.1 Case of 0-dimensional algebraic sets

Let us consider the algebra $A_F = \frac{\mathbb{R}[x_1,\ldots,x_n]}{(F_1,\ldots,F_k)}$ and let us assume that

$$\dim_{\mathbb{R}} A_F < +\infty.$$

This implies that $W$ is a finite number of points and that $\chi(W) = \#W$.

For each $f \in A_F$, let $L_f : A_F \to A_F$ be the linear endomorphism of multiplication by $f$, i.e. $\forall g \in A_F$, $L_f(g) = fg$. Let $Q$ be the quadratic form $A_F$ defined by $Q(f) = \text{Trace}(L_{f^2})$.

**Theorem 4.1.1** *([4], [31]) We have*

$$\text{signature } Q = \#W = \chi(W).$$

## 4.2 Case of a compact algebraic set

Here we present a formula due to Bruce [6] and Szafraniec [32], [35] that expresses the Euler characteristic of a compact algebraic set of $\mathbb{R}^n$ as a signature of a quadratic form. It relies strongly on the formula for the Euler characteristic of the link of a homogeneous singularity (Proposition 3.3.2).

We recall that $W = F_1^{-1}(0) \cap \cdots \cap F_k^{-1}(0)$ and we assume that $W$ is compact. Let $d$ be the highest degree of the $F_i$'s. For $i = 1, \ldots, k$, let $G_i$ be the homogeneous polynomial function in the variables $x_0, x_1, \ldots, x_n$ of degree $d+1$ defined by

$$G_i(x_0, x_1, \ldots, x_n) = x_0^{d+1} F_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Let $G$ be the polynomial function given by $G = G_1^2 + \cdots + G_k^2$. Then $G$ is homogeneous of degree $2d+2$. Let $k = 2d+4$ and let $\bar{G} : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by

$$\bar{G}(x_0, x_1, \ldots, x_n) = G(x_0, x_1, \ldots, x_n) - \frac{x_0^k + x_1^k + \cdots + x_n^k}{k}.$$

**Theorem 4.2.1** *The polynomial function $\bar{G}$ has an algebraically isolated critical point at the origin and*

$$\chi(W) = \frac{1}{2}\left((-1)^n - \mathrm{Ind}(\nabla \bar{G}, 0)\right).$$

*Proof.* Since the $G_i$'s are homogeneous, $W$ is homeomorphic to the sets

$$\{x \in S^n \mid x_0 > 0, G_1(x) = \cdots = G_k(x) = 0\},$$

and

$$\{x \in S^n \mid x_0 < 0, G_1(x) = \cdots = G_k(x) = 0\}.$$

Since $W$ is compact,

$$\chi\left(\{x \in S^n \mid G_1(x) = \cdots = G_k(x) = 0\}\right) = 2\chi(W) + \chi(S^{n-1}).$$

So we get that

$$\chi(W) = \frac{1}{2}\left(\chi\left(\{x \in S^n \mid G_1(x) = \cdots = G_k(x) = 0\}\right) - 1 - (-1)^{n-1}\right) = \frac{1}{2}\left(\chi\left(\{x \in S^n \mid G(x) \leq 0\}\right) - 1 + (-1)^n\right),$$

and, by Proposition 3.3.2,

$$\chi(W) = \frac{1}{2}\left(1 - \mathrm{Ind}(\nabla \bar{G}, 0) - 1 + (-1)^n\right).$$

□

**Remark 4.2.2** *1. As observed by Zbigniew Szafraniec, if $W$ is not compact, the formula is still valid replacing $\chi(W)$ by $\chi_c(W)$.*

*2. Since $\bar{G}$ has an algebraically isolated critical point at 0, $\mathrm{Ind}(\nabla \bar{G}, 0)$ can be computed as the signature of a quadratic form on*

$$\frac{\mathbb{R}\{x_0, x_1, \ldots, x_n\}}{\left(\frac{\partial \bar{G}}{\partial x_0}, \frac{\partial \bar{G}}{\partial x_1}, \ldots, \frac{\partial \bar{G}}{\partial x_n}\right)},$$

*thanks to the Eisenbud-Levine formula.*

## 4.3 Case of a smooth complete intersection

The above formula works only for compact algebraic sets of $\mathbb{R}^n$. When $W$ is not compact, Bruce and Szafraniec also obtained a signature formula for $\chi(W)$, but it is not effective since one can not compute explicitely one of the constant involved in the formula.

In this section, we present a method due to Szafraniec [33] in the case where $W$ is a smooth complete intersection of codimension $k$. We recall that $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ is a polynomial mapping and that $W = F^{-1}(0)$. Furthermore, we assume that $1 \leq k \leq n-1$ and that for $x \in W$, $\mathrm{rank}[DF(x)] = k$. This implies that $W$ is a smooth manifold of dimension $n-k$ (maybe empty). Therefore, we can use tools of differential topology to study the topology of $W$.

Let $(x; \lambda) = (x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k)$ be a coordinate system of $\mathbb{R}^{n+k}$. Let $H$ be the following polynomial mapping:

$$\begin{array}{rccc} H & : & \mathbb{R}^{n+k} & \to & \mathbb{R}^{n+k} \\ & & (x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k) & \mapsto & (x + \sum_{i=1}^k \lambda_i \nabla F_i(x), F(x)). \end{array}$$

**Theorem 4.3.1** *The set $H^{-1}(0)$ is compact. Let $R > 0$ be such that $H^{-1}(0) \subset B_R^{n+k}$ then*

$$\chi(W) = (-1)^k \mathrm{Ind}(H, \infty),$$

*where $\mathrm{Ind}(H, \infty)$ is the Brouwer degree of the map $\frac{H}{\|H\|} : S_R^{n+k-1} \to S^{n+k-1}$.*

*Proof.* We just give a sketch of proof. The reader can refer back to Szafraniec's paper for the details. The fact that $H^{-1}(0)$ is compact is not difficult to prove. For simplicity, we assume that $H$ has only non-degenerate roots. This means that for each $(p,\lambda) \in H^{-1}(0)$, $\det[DH(p,\lambda)] \neq 0$, where $DH$ is the Jacobian matrix of $H$. Let us write

$$H^{-1}(0) = \{(p_1,\lambda_1),\ldots,(p_r,\lambda_r)\}.$$

Let $\rho(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$, then $\nabla\rho(x) = x$ for $x \in \mathbb{R}^n$. The points $p_i$ are exactly the critical points of $\rho_{|W}$. Furthermore they are Morse critical points and, if $s_i$ is the Morse index of $\rho_{|W}$ at $p_i$ then, by a computation of determinants, we get

$$(-1)^{s_i} = (-1)^k \text{sign } \det[DH(p_i,\lambda_i)].$$

We can apply Morse theory to $\rho_{|W}$ and we obtain that

$$\chi(W) = \sum_{i=1}^{r}(-1)^{s_i} = \sum_{i=1}^{r}(-1)^k \text{sign } \det[DH(p_i,\lambda_i)] = (-1)^k\left(\sum_{i=1}^{r}\text{sign } \det[DH(p_i,\lambda_i)]\right).$$

But by the application after Theorem 2.1.7, we know that

$$\text{Ind}(H,\infty) = \sum_{i=1}^{r}\text{sign } \det[DH(p_i,\lambda_i)].$$

□

The formula in the above theorem is completely explicit and if

$$\dim_{\mathbb{R}}\frac{\mathbb{R}[x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_k]}{\langle H\rangle} < +\infty,$$

where $\langle H\rangle$ is the ideal generated by the components of $H$, then $\text{Ind}(H,\infty)$ can be computed as the signature of a quadratic form on this polynomial algebra (see [37]). However, we have to add $k$ more variables and this makes the computations difficult in practice. In the sequel, we will briefly present a second method due to Szafraniec and his collaborators which is more algebraic and more effective, because it does not require any extra variable.

Let $A$ be the following algebra:

$$A = \frac{\mathbb{R}[x_1,\ldots,x_n]}{\left(F_1,\ldots,F_k;\frac{\partial(\rho,F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}\right)},$$

where $\frac{\partial(\rho,F_1,\ldots,F_k)}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}$ is the $(k+1)\times(k+1)$-minor of the Jacobian matrix $D(\rho,F_1,\ldots,F_k)$ with respect to the variables $x_{i_1},\ldots,x_{i_{k+1}}$. We assume that $A$ is a finite dimensional real vector space. The following theorem was proved in [8].

**Theorem 4.3.2** *We have*

$$\chi(W)\equiv \dim_{\mathbb{R}} A \bmod 2.$$

This result was later improved by Szafraniec in [36]. He managed to construct a linear functional $\varphi$ on $A$ and with this linear functional, he defined two bilinear symmetric forms $\Phi$ and $\Phi_M$ in the following way:

$$\forall (f,g)\in A\times A,\quad \Phi(f,g)=\varphi(fg) \text{ and } \Phi_M(f,g)=\varphi(Mfg),$$

where

$$M=\frac{\partial(F_1,\ldots,F_k)}{\partial(x_1,\ldots,x_k)}.$$

The following theorem relates $\chi(W)$ to the signatures of $\Phi$ and $\Phi_M$.

**Theorem 4.3.3** *If $n-k$ is odd then we have*

$$\chi(W)=(-1)^k \text{signature } \Phi.$$

*If $n-k$ is even then we have*

$$\chi(W)=\text{signature } \Phi_M.$$

## 4.4 Case of complete intersections with isolated singularities

In two papers [11, 14], we improved the above results of Szafraniec in some cases where $W$ has isolated singularities.

In [11], we did not assume that $W$ was smooth but we still assumed that the dimension of the above algebra $A$ was finite. This implies that $W$ can admit a finite number of singularities that we denote by $q_1,\ldots,q_r$.

**Theorem 4.4.1** *We have*

$$\chi(W)+\sum_{j=1}^{r}\mu(F_{\mathbb{C}},q_j)\equiv \dim_{\mathbb{R}} A \bmod 2,$$

*where $F_{\mathbb{C}}$ is the complexification of $F$ and $\mu(F_{\mathbb{C}},q_j)$ is its Milnor number at $q_j$.*

In the same paper, we were able to replace this mod 2 equality with an equality in $\mathbb{Z}$ in the case of curves ($k = n-1$) ([11], Theorem 5.2, Corollary 5.3 and Theorem 5.4) and for odd dimensional hypersurfaces ($n$ even, $k = 1$) ([11], Theorem 4.3 and Corollary 4.4).

Later in [14], we found a new method that works for all hypersurfaces, even or odd dimensional ones. Let us explain it now.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function such that its set of critical points $\nabla F^{-1}(0)$ is finite and such that $F(0) > 0$. This implies that $W = F^{-1}(0)$ is an hypersurface with isolated singularities and that $0 \notin W$. Let $(x, \lambda) = (x_1, \ldots, x_n, \lambda)$ be a coordinate system of $\mathbb{R}^{n+1}$ and let us define four polynomial mappings $H$, $K$, $L_1$ and $L_2$ as follows:

$$H(x,\lambda) = (\lambda x + \nabla F, F),\ K(x,\lambda) = (\lambda x + \nabla F, \lambda F),\ L_1(x,\lambda) = (\nabla F, \lambda F - 1),$$

$$\text{and } L_2(x,\lambda) = (\nabla F, \lambda F^2 - 1).$$

The next theorem can be viewed as a global version of the Khimshiashvili formula (Theorem 3.2.1).

**Theorem 4.4.2** *([14], Theorem 5.10). The sets $H^{-1}(0)$, $K^{-1}(0)$, $L_1^{-1}(0)$ and $L_2^{-1}(0)$ are compact. If $n$ is even, then*

$$\chi(W) = \mathrm{Ind}(H, \infty) + \mathrm{Ind}(\nabla F, \infty) - \mathrm{Ind}(L_2, \infty),$$

$$\chi(\{F \geq 0\}) - \chi(\{F \leq 0\}) = 1 - \mathrm{Ind}(K, \infty) + \mathrm{Ind}(L_1, \infty).$$

*If $n$ is odd, then*

$$\chi(W) = \mathrm{Ind}(K, \infty) - \mathrm{Ind}(L_1, \infty),$$

$$\chi(\{F \geq 0\}) - \chi(\{F \leq 0\}) = 1 - \mathrm{Ind}(H, \infty) - \mathrm{Ind}(\nabla F, \infty) + \mathrm{Ind}(L_2, \infty).$$

*Proof.* We will just prove briefly the equality

$$\chi(W) = \mathrm{Ind}(K, \infty) - \mathrm{Ind}(L_1, \infty),$$

when $n$ is odd. The fact that $K^{-1}(0)$ and $L_1^{-1}(0)$ are compact is easy to prove. Let us write

$$\nabla F^{-1}(0) = \{q_1, \ldots, q_t\} = \{q_1, \ldots, q_r\} \cup \{q_{r+1}, \ldots, q_t\},$$

where for $i \in \{1, \ldots, r\}$, $q_i \in W$ and for $j \in \{r+1, \ldots, t\}$, $q_j \notin W$.

Let us choose $R' \gg 1$ sufficiently big so that $\chi(W) = \chi(W \cap B^n_{R'})$ and $\nabla F^{-1}(0) \subset B^n_{R'}$. Let $\delta$, $0 < |\delta| \ll 1$, be a small regular value of $F$. The

following equality is proved in the same way as the Khimshiashvili formula (Theorem 3.2.1):

$$\chi(F^{-1}(\delta) \cap B^n_{R'}) = \chi(W \cap B^n_{R'}) + (\text{sign } \delta) \sum_{i=1}^{r} \text{Ind}(\nabla F, q_i).$$

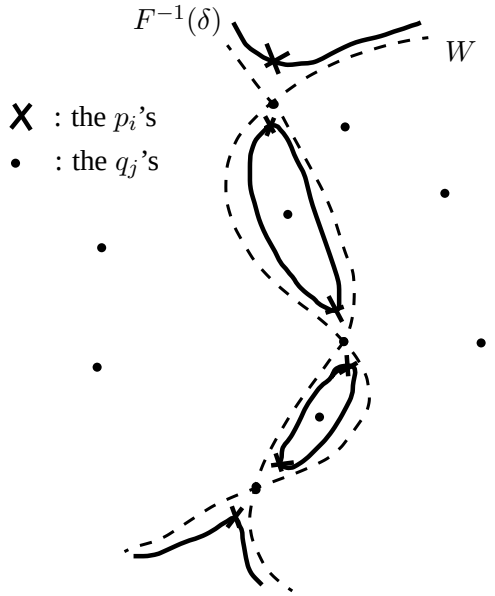
Let $K_\delta$ be the following perturbation of $K$:

$$K_\delta(x, \lambda) = (\lambda x + \nabla F, \lambda(F - \delta)).$$

Let $R > 0$ be such that $K^{-1}(0) \subset B^{n+1}_R$. If $R$ is sufficiently big and $\delta$ sufficiently small then the mapping $\frac{K}{\|K\|} : S^n_R \to S^n$ and $\frac{K_\delta}{\|K_\delta\|} : S^n_R \to S^n$ are homotopic. This implies that $\text{Ind}(K, \infty)$ is equal to the topological degree of the map $\frac{K_\delta}{\|K_\delta\|} : S^n_R \to S^n$.

Let $\rho(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. Then it is not difficult to see that

$$K_\delta^{-1}(0) \cap B^{n+1}_R = \left\{(p, \lambda) \mid p \text{ critical point of } \rho_{|F^{-1}(\delta) \cap B^n_{R'}}\right\} \bigcup \sqcup_{j=1}^{t} \{(q_j, 0)\}.$$




For simplicity, let us assume that $\rho_{|F^{-1}(\delta) \cap B^n_{R'}}$ is a Morse function. Then it admits a finite number of critical points $p_1, \ldots, p_m$, with respective indices $s_1, \ldots, s_m$, and

$$K_\delta^{-1}(0) \cap B^{n+1}_R = \{(p_1, \lambda_1), \ldots, (p_m, \lambda_m)\} \bigcup \sqcup_{j=1}^{t} \{(q_j, 0)\}.$$

A computation of determinants gives that

$$(-1)^{s_i} = \text{sign}\left[(\lambda_i)^{n-1}\det[DK_\delta(p_i,\lambda_i)]\right],$$

for $i = 1, \ldots, m$. By Morse theory and since $n$ is odd, we have

$$\chi(F^{-1}(\delta) \cap B^n_{R'}) = \sum_{i=1}^{m}(-1)^{s_i} = \sum_{i=1}^{m} \text{ sign } \det[DK_\delta(p_i,\lambda_i)].$$

It remains to study $\text{Ind}(K_\delta, (q_j, 0))$ for $j = 1, \ldots, t$. It is not difficult to see that

$$\text{Ind}(K_\delta, (q_j, 0)) = \text{sign}(F(q_j) - \delta)\text{Ind}(\nabla F, q_j).$$

Actually, this equality is easy to prove when $q_j$ is a non-degenerate zero of $\nabla F$. To establish it in general case, we just use a small perturbation of $F$. Putting together all these results, we obtain that

$$\text{Ind}(K, \infty) = \chi(F^{-1}(\delta) \cap B^n_{R'}) + \sum_{j=1}^{t}\text{sign}(F(q_j) - \delta)\text{Ind}(\nabla F, q_j) =$$

$$\chi(F^{-1}(\delta) \cap B^n_{R'}) + \sum_{j=1}^{r}\text{sign}(F(q_j) - \delta)\text{Ind}(\nabla F, q_j) +$$

$$\sum_{j=r+1}^{t} \text{sign}(F(q_j) - \delta)\text{Ind}(\nabla F, q_j).$$

For $j = 1, \ldots, r$, we have $\text{sign}(F(q_j) - \delta) = -\text{sign}(\delta)$ and for $j = r+1, \ldots, t$, $\text{sign}(F(q_j) - \delta) = \text{sign}(F(q_j))$ because $\delta$ is a small regular of $f$. Therefore, we get

$$\text{Ind}(K, \infty) = \chi(F^{-1}(\delta) \cap B^n_{R'}) - \text{sign}(\delta)\sum_{j=1}^{r}\text{Ind}(\nabla F, q_j) +$$

$$\sum_{j=r+1}^{t} \text{sign}(F(q_j))\text{Ind}(\nabla F, q_j) = \chi(W) + \sum_{j=r+1}^{t} \text{sign}(F(q_j))\text{Ind}(\nabla F, q_j).$$

But the zero set of $L_1$ is exactly $\cup_{j=r+1}^{t}(q_j, \frac{1}{F(q_j)})$ and a straightforward calculation shows that

$$\text{Ind}\left(L_1, (q_j, \frac{1}{F(q_j)})\right) = \text{sign}(F(q_j))\text{Ind}(\nabla F, q_j).$$

□

Let us end this section and this chapter with some remarks.

**Remark 4.4.3** *1. In [15], we generalized Szafraniec's results (Theorem 4.3.1) to the case when $W$ admits a compact singular set, i.e. the set*

$$\{x \in W \mid \operatorname{rank}[DF(x)] < k\},$$

*is compact. However, although the formula is completely explicit, it is not effective in practice because we need to add $k+1$ variables and we use a Łojasiewicz exponent of size $O(2kd)^n$, where $d$ is the degree of the map $F$.*

*2. As in Theorem 4.4.2, we can also compute the Euler characteristic of semi-algebraic sets instead of algebraic sets. That is what we did in the papers [10, 12, 13], where we proved several semi-algebraic versions of Proposition 3.3.2 and Theorems 4.2.1 and 4.3.1.*

*The PhD thesis of Julie Lapébie [25] contains also several semi-algebraic versions of Theorem 4.3.1 and Theorem 4.4.2.*

# Chapter 5

# Topology of the real Milnor fibration

In this chapter, we review some results about the real Milnor fibration and the topology of the real Milnor fibre. We start with some results due to Massey [27] which generalize the ones of Milnor [30], Chapter 11.

## 5.1 Milnor's conditions (a) and (b) and the fibration theorem

Let $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be an analytic map-germ, $1 \leq k \leq n-1$, $V = F^{-1}(0)$ and $\Sigma_F$ be the set of critical points of $F$, i.e.

$$\Sigma_F = \{x \in \mathbb{R}^n \mid \operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_k(x)) < k\},$$

(of course $F$ is not constant).

Let $\rho(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$ and let us denote by $\Sigma_{F,\rho}$ the set of critical points of the pair $(F, \rho)$, i.e.

$$\Sigma_{F,\rho} = \{x \in \mathbb{R}^n \mid \operatorname{rank}(\nabla f_1(x), \ldots, \nabla f_k(x), \nabla \rho(x)) < k+1\}.$$

It is clear that $\Sigma_F \subseteq \Sigma_{F,\rho}$.

The following two conditions were introduced by Massey [27].

**Definition 5.1.1** *[27] Let $F$ and $\rho$ be as above.*

1. *We say that $F$ satisfies Milnor's condition $(a)$ at the origin if $\Sigma_F \subset V$ in a neighborhood of the origin.*

2. *We say that $F$ satisfies Milnor's condition (b) at the origin if* 0 *is isolated in* $V \cap \overline{\Sigma_{F,\rho} \setminus V}$ *in a neighborhood of the origin.*

**Example 5.1.2** *1. Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a holomorphic function-germ. It is well known that* $\Sigma_f \subset f^{-1}(0)$ *in a neighborhood of the origin so it satisfies Milnor's conditions* $(a)$.

*Furthermore, Hamm and Lê [21], page 323, proved that the Thom* $a_f$*-condition is satisfied for a Whitney stratification of* $V$. *Hence, Milnor's condition* $(b)$ *is also satisfied.*

2. *Similarly if* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is an analytic function-germ then it satisfies Milnor's conditions* $(a)$ *and* $(b)$. *Condition* $(b)$ *follows from the fact that the* $a_f$*-condition is satisfied (see [5], Proposition 10).*

3. *Let* $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ *be an analytic map-germ with an isolated singular point at origin. Then Milnor's conditions* $(a)$ *and* $(b)$ *hold. Milnor's condition* $(b)$ *is satisfied because the zero locus* $V$ *of* $F$ *is transverse to all small spheres.*

We say that $\epsilon > 0$ is a *Milnor radius for* $F$ *at the origin* if

$$B^n_\epsilon \cap (\overline{\Sigma_F - V}) = \emptyset \text{ and } B^n_\epsilon \cap V \cap (\overline{\Sigma_{F,\rho} \setminus V}) \subseteq \{0\}.$$

Under Milnor's conditions $(a)$ and $(b)$, we see that for $0 < |\delta| \ll \epsilon \ll 1$, the fibre $F^{-1}(\delta)$ is smooth and transverse to $S^{n-1}_\epsilon$.

**Theorem 5.1.3** *([27], page 284, Theorem 4.3) Let* $F : (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $2 \leq k \leq n-1$, *be an analytic map-germ satisfying Milnor's conditions* $(a)$ *and* $(b)$ *at the origin and let* $\epsilon_0 > 0$ *be a Milnor radius for* $F$ *at origin. Then, for each* $0 < \epsilon \leq \epsilon_0$, *there exists* $\delta \in \mathbb{R}^k$ *with* $0 < |\delta| \ll \epsilon$, *such that*

$$\bar{F} : B^n_\epsilon \cap F^{-1}(B^k_\delta \setminus \{0\}) \to B^k_\delta \setminus \{0\}, \tag{5.1}$$

*where* $\bar{F}$ *is the restriction of* $F$ *to* $B^n_\epsilon \cap F^{-1}(B^k_\delta \setminus \{0\})$, *is the projection of a smooth locally trivial fibration.*

*Proof.* It relies on Ehresmann's fibration theorem for manifolds with boundary. It is proved in [27], Theorem 4.3 in the general case and in [30], Theorem 11.2 and Lemma 11.3 in the case of an isolated critical point. □

From now on, we will denote by $M_F$ the fibre of this fibration.

**Example 5.1.4** *This example is inspired by examples in [7].*

*Let $F : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be defined by $F(x, y, z) = (x^2z + y^2, x)$. It is easy to see that $\Sigma_F = \{(0, 0, z) : z \in \mathbb{R}\} \subseteq V$ and so Milnor's condition $(a)$ holds. However, for any $\delta > 0$ the fibres $F^{-1}(\delta, 0)$ and $F^{-1}(-\delta, 0)$ are not homeomorphic because $F^{-1}(-\delta, 0)$ is empty. Hence, Milnor's condition $(b)$ does not hold.*

## 5.2 Topology of the real Milnor fibre

In this section, we give a method to obtain a topological degree formula for the Euler characteristic of $M_F$. We still consider $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $1 \leq k \leq n-1$, an analytic map-germ. Let $l \in \{1, \ldots, k\}$ and $I = \{i_1, \ldots, i_l\}$ be an $l$-tuple of pairwise distinct elements of $\{1, \ldots, k\}$. We denote by $f_I$ the mapping $(f_{i_1}, \ldots, f_{i_l}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$. Suppose that $F$ satisfies Milnor condition's $(a)$ at the origin. Then, we have

$$\Sigma_{f_I} \subset \Sigma_F \subset F^{-1}(0) \subset f_I^{-1}(0),$$

and so the map $f_I$ also satisfies Milnor's condition $(a)$ at the origin.

The following results are proved in [16], Section 4.

**Lemma 5.2.1** *Assume that $F$ satisfies Milnor's conditions $(a)$ and $(b)$ at the origin. Then, for $l \in \{1, \ldots, k\}$ and $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, the map $f_I : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$ satisfies Milnor's conditions $(a)$ and $(b)$.*

**Corollary 5.2.2** *There exists $\epsilon_0 > 0$ such that, for all $l \in \{1, \ldots, k\}$ and $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, the maps $f_I : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$ have $\epsilon_0$ as a Milnor radius.*

Now we give a result which was stated as a conjecture in Milnor [30], Chapter 11. It was first proved by King in his unpublished PhD-Thesis in the isolated singularity case and by the author and Araújo dos Santos [16] in the general case. We note that Jacquemard [23] also proved a version of this Milnor conjecture.

**Theorem 5.2.3** *(Milnor's conjecture) Let $F : (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $3 \leq k \leq n-1$, be an analytic map-germ that satisfies Milnor's conditions (a) and (b) at the origin and let $\phi$ be the mapping $(f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)$. Let $M_F$ be the Milnor fibre of $F$ and let $M_\phi$ be the Milnor fibre of $\phi$. Then the fibre $M_\phi$ is homeomorphic to $M_F \times [-1, 1]$.*

*Proof.* We just give an idea of the proof for an isolated singularity. We denote by $g$ the function $f_k$. By an easy application of the Curve Selection Lemma, we can prove that for $\epsilon > 0$ sufficiently small, the critical points of $g_{|\phi^{-1}(0)\cap S_\epsilon^{n-1}}$ lie in $\{g \neq 0\}$ (see [16], Lemma 3.1) and are outwards-pointing in $\{g > 0\}$ and inwards-pointing in $\{g < 0\}$ ([16], Lemma 3.3). So let $\epsilon > 0$ be a Milnor radius for $F$ and $\phi$ and let $\delta \in \mathbb{R}^{k-1}$ be a regular value of $\phi$. We can assume that

1. $0 < \|\delta\| \ll \epsilon \ll 1$,
2. $M_\phi$ is homeomorphic to $\phi^{-1}(\delta) \cap B_\epsilon^n$,
3. $M_F$ is homeomorphic to $\phi^{-1}(\delta) \cap g^{-1}(0) \cap B_\epsilon^n$ because $\Sigma_F \subset F^{-1}(0)$,
4. the critical points of $g$ restricted to $\phi^{-1}(\delta) \cap B_\epsilon^n$ lie in $\{g \neq 0\}$, are outwards-pointing in $\{g > 0\}$ and inwards-pointing in $\{g < 0\}$.

Note that $g_{|\phi^{-1}(\delta)\cap \mathring{B}_\epsilon^n}$ has no critical points because $\Sigma_{\phi,g} \subset \phi^{-1}(0) \cap g^{-1}(0)$. By a version of Ehresmann's fibration theorem for manifolds with boundary (see for instance [16], Lemma 6.1), $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g \geq 0\}$ is homeomorphic to $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g = 0\} \times [0, 1]$. Similarly $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g \leq 0\}$ is homeomorphic to $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g = 0\} \times [-1, 0]$. Therefore $\phi^{-1}(\delta) \cap B_\epsilon^n$ is homeomorphic to $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g = 0\} \times [-1, 1]$ because it is homeomorphic to the gluing of $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g \geq 0\}$ and $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g \leq 0\}$ along $\phi^{-1}(\delta) \cap B_\epsilon^n \cap \{g = 0\}$. □

**Corollary 5.2.4** *Under the above conditions, we have* $\chi(M_F) = \chi(M_\phi)$.

**Corollary 5.2.5** *Let* $l \in \{2, \ldots, k\}$ *and let* $I = \{i_1, \ldots, i_l\}$ *be an* $l$*-tuple of pairwise distinct elements of* $\{1, \ldots, k\}$. *Then we have* $\chi(M_{f_I}) = \chi(M_F)$.

It remains to consider the fibres of the function $f_j : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Here such a function admits two Milnor fibres : $M_{f_{\{j\}}}^+ = f_j^{-1}(\delta) \cap B_\epsilon^n$ and $M_{f_{\{j\}}}^- = f_j^{-1}(-\delta) \cap B_\epsilon^n$, where $0 < \delta \ll \epsilon \ll 1$.

Let us write for instance $f = f_1$ and $g = f_2$. Using the same argument as above, we see that $M_f^+$ is homeomorphic to $M_{(f,g)} \times [-1, 1]$ and that $M_f^-$ is also homeomorphic $M_{(f,g)} \times [-1, 1]$.

**Corollary 5.2.6** *For every* $j \in \{1, \ldots, k\}$, *we have* $\chi(M_{f_{\{j\}}}^+) = \chi(M_{f_{\{j\}}}^-) = \chi(M_F)$.

In the case of an isolated singularity, we can easily deduce from this corollary and the Khimshiashvili formula a topological degree formula for $\chi(M_F)$.

**Proposition 5.2.7** *Let $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $1 \leq k \leq n-1$, be an analytic map-germ with an isolated singularity at the origin. Then for $i = 1, \ldots, k$, $f_i$ also has an isolated singularity at the origin. The following holds:*

*(i) if $n$ is even, then* $\mathrm{Ind}(\nabla f_1, 0) = \cdots = \mathrm{Ind}(\nabla f_k, 0)$ *and*

$$\chi(M_F) = 1 - \mathrm{Ind}(\nabla f_1, 0),$$

*(ii) if $n$ is odd, then* $\mathrm{Ind}(\nabla f_1, 0) = \cdots = \mathrm{Ind}(\nabla f_k, 0) = 0$ *and*

$$\chi(M_F) = 1.$$

*Proof.* We know that for all $i \in \{1, \ldots, k\}$,

$$\chi(M_F) = \chi(M^+_{f_{\{i\}}}) = \chi(M^-_{f_{\{i\}}}).$$

If $n$ is even, by the Khimshiashvili formula,

$$\chi(M^+_{f_{\{i\}}}) = \chi(M^-_{f_{\{i\}}}) = 1 - \mathrm{Ind}(\nabla f_i, 0).$$

If $n$ is odd, we have

$$\chi(M^+_{f_{\{i\}}}) = 1 + \mathrm{Ind}(\nabla f_i, 0) = \chi(M^-_{f_{\{i\}}}) = 1 - \mathrm{Ind}(\nabla f_i, 0).$$

□

Now we would like to explain how to establish a similar formula in the case of a mapping $F$ satisfying Milnor's condition $(a)$ and $(b)$. The strategy is to relate $\chi(M_F)$ to the Euler characteristic of the link of a real analytic singularity and apply Szafraniec's result (Proposition 3.3.1).

Let $F = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $1 \leq k \leq n-1$, be an analytic map-germ satisfying Milnor's conditions $(a)$ and $(b)$ at the origin. Let us choose $l \in \{2, \ldots, k\}$ and an $l$-tuple $I = \{i_1, \ldots, i_l\}$ of pairwise distinct elements of $\{1, \ldots, k\}$. We write $J = \{i_1, \ldots, i_{l-1}\}$ and $g = f_{i_l}$. We also denote by $\mathcal{L}_I$ (resp. $\mathcal{L}_J$) the link of the zero-set of $f_I$ (resp. $f_J$). If $l = 1$ then we put $J = \emptyset$ and $f_J = 0$.

**Proposition 5.2.8** *We have:*

$$\chi(\mathcal{L}_J) - \chi(\mathcal{L}_I) = (-1)^{n-l} 2\chi(M_F).$$

*Proof.* Let us write $V_J = f_J^{-1}(0)$. By a deformation argument due to Milnor, $V_J \cap g^{-1}(\delta) \cap B_\epsilon^n$ is homeomorphic to $V_J \cap \{g \geq \delta\} \cap S_\epsilon^{n-1}$ and $V_J \cap g^{-1}(-\delta) \cap B_\epsilon^n$ is homeomorphic to $V_J \cap \{g \leq -\delta\} \cap S_\epsilon^{n-1}$ for $0 < \delta \ll \epsilon \ll 1$. By the Mayer-Vietoris sequence, we can write:

$$\chi(V_J \cap S_\epsilon^{n-1}) = \chi(V_J \cap S_\epsilon^{n-1} \cap \{g \geq \delta\}) + \chi(V_J \cap S_\epsilon^{n-1} \cap \{g \leq -\delta\}) +$$

$$\chi(V_J \cap S_\epsilon^{n-1} \cap \{-\delta \leq g \leq \delta\}) - \chi(V_J \cap S_\epsilon^{n-1} \cap \{g = \delta\}) - \chi(V_J \cap S_\epsilon^{n-1} \cap \{g = -\delta\}).$$

By the above remark and Corollaries 5.2.5 and 5.2.6, the first two terms of the right-hand side of this equality are equal to $\chi(M_F)$. The third term is equal to $\chi(\mathcal{L}_I)$ because by Durfee's result ([9], Proposition 1.6), $\mathcal{L}_I$ is a retract by deformation of $V_J \cap S_\epsilon^{n-1} \cap \{-\delta \leq g \leq \delta\}$. Furthermore, if $n - l$ is even then the last two Euler characteristics are equal to 0 because $V_J \cap S_\epsilon^{n-1} \cap \{g = \delta\}$ and $V_J \cap S_\epsilon^{n-1} \cap \{g = -\delta\}$ are odd-dimensional compact manifolds. If $n - l$ is odd, they are equal to $2\chi(M_F)$ because they are boundaries of odd-dimensional Milnor fibres of $f_I$. □

**Corollary 5.2.9** *Let $j \in \{1, \ldots, k\}$. If $n$ is even, then we have $\chi(\mathcal{L}_{\{j\}}) = 2\chi(M_F)$ and if $n$ is odd, then we have $\chi(\mathcal{L}_{\{j\}}) = 2 - 2\chi(M_F)$.*

*Proof.* We apply the previous proposition to the case $l = 1$. In this case, if $n$ is even then $\chi(\mathcal{L}_J) = 0$ and if $n$ is odd then $\chi(\mathcal{L}_J) = 2$. □

**Corollary 5.2.10** *Let $l \in \{3, \ldots, k\}$ and let $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$. Let $K$ be an $(l-2)$-tuple of pairwise distincts elements of $I$. Then we have $\chi(\mathcal{L}_K) = \chi(\mathcal{L}_I)$.*

*Proof.* Let $J$ be an $(l-1)$-tuple built from adding to $K$ one element of $I \setminus K$. By the previous proposition, we see that $\chi(\mathcal{L}_J) - \chi(\mathcal{L}_I) = \chi(\mathcal{L}_J) - \chi(\mathcal{L}_K)$. □

So, in order to express the Euler characteristics of all the links $\mathcal{L}_I$, we just need to compute the Euler characteristic of a link $\mathcal{L}_I$ where $\#I = 2$. Let us set $I = \{1, 2\}$. By Proposition 5.2.8, we find that

$$\chi(\mathcal{L}_I) = \chi(\mathcal{L}_{\{1\}}) - (-1)^n 2\chi(M_F).$$

So we see that $\chi(\mathcal{L}_I) = 0$ if $n$ is even and that $\chi(\mathcal{L}_I) = 2$ if $n$ is odd. We can summarize all these results in the following theorem.

**Theorem 5.2.11** *Let $l \in \{1, \ldots, k\}$ and let $I = \{i_1, \ldots, i_l\}$ be an $l$-tuple of pairwise distinct elements of $\{1, \ldots, k\}$. If $n$ is even, then we have*

$$\chi(\mathcal{L}_I) = 2\chi(M_F) \text{ if } l \text{ is odd and } \chi(\mathcal{L}_I) = 0 \text{ if } l \text{ is even.}$$

*If $n$ is odd, then we have*

$$\chi(\mathcal{L}_I) = 2 - 2\chi(M_F) \text{ if } l \text{ is odd and } \chi(\mathcal{L}_I) = 2 \text{ if } l \text{ is even.}$$

Using this last theorem and Szafraniec's formula (Proposition 3.3.1), we can state a Poincaré-Hopf type formula for non-isolated singularities.

**Proposition 5.2.12** *Let $F : (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $1 \leq k \leq n-1$, be an analytic map-germ that satisfies Milnor's conditions (a) and (b) at the origin. Then there exist $c > 0$ and an integer $K$ such that for $g(x) = f_1^2(x) - c\rho(x)^K$, we have*

*(i) if $n$ is even, then $\chi(M_F) = \frac{1}{2}(1 - \mathrm{Ind}(\nabla g, 0))$,*

*(ii) if $n$ is odd, then $\chi(M_F) = \frac{1}{2}(1 + \mathrm{Ind}(\nabla g, 0))$.*

Of course if one of the components of $F$ is homogeneous, we can use the Bruce-Szafraniec formula (Proposition 3.3.2) to get a signature for $\chi(M_F)$.

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